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GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

PROGRESS REPORT NO. 3 ON STUDIES IN THE FIELDS OF SPACE FLIGHT AND GUIDANCE THEORY

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AEROBALLISTICS DIVISION
MARSHALL SPACE FLIGHT CENTER,
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GEORGE C. MARSHALL SPACE FLIGHT CENTER

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PROGRESS REPORT NO. 3

on Studies in the Fields of

SPACE FLIGHT AND GUIDANCE THEORY

Sponsored by Aeroballistics Division of

Marshall Space Flight Center

ABSTRACT

This paper contains progress reports of NASA-sponsored studies in the areas of space flight theory and guidance theory. The studies are carried on by several universities and industrial companies. This progress report covers the period from June 15, 1962 to December 20, 1962. The technical supervisor of the contracts is W. E. Miner, Deputy Chief of the Future Projects Branch of Aeroballistics Division, George C. Marshall Space Flight Center.

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SUMMARY

This paper contains progress reports of NASA-sponsored studies in the areas of space flight theory and guidance theory. The studies are carried on by several universities and industrial companies. This progress report covers the period from June 15, 1962 to December 20, 1962. The technical supervisor of the contracts is W. E. Miner, Deputy Chief of the Future Projects Branch of Aeroballistics Division, George C. Marshall Space Flight Center (MSFC).

INTRODUCTION

Eleven papers are collected in this report. These papers have been written by research investigators employed at agencies under contract to MSFC. The subject matter lies in the areas of guidance theory and space flight theory.

This report is the third of the "Progress Reports" and covers the period from June 15, 1962 to December 20, 1962. This progress report will hereinafter be referred to as "the report." Progress Report No. 1 (2) will be referred to as the "first (second) report." Information given in the first and second reports is not repeated in this report.

The agencies contributing and their fields of major interest are:

Field of Interest	Agency
Calculus of Variations	Vanderbilt University Republic Aviation Corporation Grumman Aircraft Engineering Co. Hayes International Corporation General Electric Company
Impulse Orbit Transfer	North American Aviation, Inc.
Celestial Mechanics	Republic Aviation Corporation
Large Computer Exploitation	Northeast Louisiana College Chrysler Corporation University of North Carolina

The objectives of this introduction are (1) to review and summarize each agency's contribution to the report and (2) to review the status of each discipline and its application to the implementation of the adaptive guidance mode. The latter review may be taken as a statement of policy.

The first paper is written by Dr. M. G. Boyce of Vanderbilt University. An application of the theory of the calculus of variations is made to our problem. This paper and the note by Dr. R. W. Hunt give enough necessary conditions for an optimum trajectory to guarantee sufficiency. The paper represents the point of particular interest that the ratios of the Lagrange multipliers may be treated as continuous over staging points. The paper is well written and most readable. It is recommended for all.

The second paper is written by Jack Richland of Republic Aviation Corporation. A system of equations (deck) for computing "optimum" trajectories is presented. Care must be taken to identify all assumptions made by the author. The deck is based on fixed end points. The two control variables are thrust direction and thrust magnitude. The mass at injection (final cutoff) is maximized. The deck permits the design of trajectories for vehicles with restartable engines. It may be used for fuel loadings in given stages. A differential correction method is used to establish initial values of the Lagrange multipliers.

The method should work well with good initial guesses. The fixed end-point assumption is not overly restrictive. However, many problems require functional relationships for injection conditions. In this case the fixed end point solution requires added optimizations. There are computational advantages to be gained by not using the assumption. Because transversality conditions may be derived from injection conditions, these conditions assure an optimum final answer. The differential correction method described in the report may be used with functionally defined injection conditions. The transversality conditions, including those for variation in burning time and mass, should be used to adjust $[\Phi\ (t_F)\ ]$ as is discussed on page 47.

The fourth and fifth papers present methods for computing low-thrust trajectories in the neighborhood of a large attracting body. The fourth paper, by Gordon Pinkham of Grumman Aircraft Engineering Corporation, presents formulations for direct methods of the calculus of variations. The fifth paper, by Harry Passmore of Hayes International Corporation, presents formulations for classical methods of the calculus of variations. The parameters used in both papers are basically the same. The differences are in the way the parameters are handled. Pinkham develops the differential equations for the parameters. The Euler-Lagrange equations for the problem are also found. These are given as equations 1, 2, and 3 along with equations 5. Equations 5 and 6 give the basic equations for the gradient procedure. This procedure has been checked for feasibility. The potential is considered good. The Passmore paper takes as a base the planetary theory of Lagrange. The vehicle is taken as one planet. The thrust action is taken as representing a second planet. The coupling effect is the perturbation for each planet. The perturbation differential equations are then developed in standard form. These will be integrated and the results will be analyzed before further development is attempted.

The next paper, by Carlos R. Cavoti of General Electric Company presents a problem with a restricted control variable. The assumptions are: (1) The rocket is always in vertical climb, (2) the thrust magnitude, bounded from above and below, is the control variable, and (3) the non-potential forces considered are those of thrust and drag. Drag is considered to be a function of velocity and altitude. The paper develops methods of finding corners, i. e. , the points at which the thrust gradient varies. A problem is used to illustrate the procedure.

The paper by Dr. D. F. Bender of North American Aviation covers a special field in optimization. Here only impulsive forces are considered. This study follows extensive work on orbital transfer. It may also be considered as an early investigation on rendezvous. This work and the techniques developed are needed for early trajectory design.

In the field of the calculus of variation, the studies in theory for high thrust trajectory problems are sufficient for our present needs. Extended effort needs to be expended in three other areas, however. The first of these areas is that of low thrust trajectory calculations. The second area is in the development of specific decks for special problems. The third area is in developing methods for expanding the λ values at $t_o(t)$. These expansions will be in terms of the stated variables at $t_o(t)$ and the desired end conditions (including motor characteristics).

Two papers in this report cover the area of optimum low thrust trajectories. Both papers present feasible methods. Grumman Aircraft Engineering Corporation and Hayes International Corporation will continue to work in these fields .

Auburn University has been developing decks for specific problems. They will continue to do this type of work. Such work requires an eye for mathematics, physics, and hardware. Other contractors may be added to assist in these studies as the need arises.

The third area is the least explored. There are two potential methods of making the expansions for $\lambda_j(t_o)$. The first is numerical; the second is analytical.

An outline of an approach for numerical expansion is given below. Reference is made to the comments on the Republic Aviation paper. The first partial derivatives have been developed in a manner similar to the one described. The transversality conditions are related to the Euler-Lagrange equations. The Jacobi equations are of one higher order. There are conditions (say, second difference transversality conditions) related to the Jacobi equations which, when combined with the third partial relations, (say, second Jacobi equations) give the second partial derivatives. Higher partial derivatives may be developed in a similar manner. Dr. R. W. Hunt has developed the second transversality conditions. This work should proceed with accuracy and speed. The differential generator developed by the University of North Carolina will be used to develop the partials. This will be done at Marshall Space Flight Center. The contributors will develop the operations needed for use of these differentials only. The steps needed in the development and the contractors involved are given below:

Step	Development	Contractor
1	1st transversality condition	completed
2	1st Jacobi equations	completed
3	2nd transversality condition	Dr. Hunt (SIU)
4	2nd Jacobi equations	Dr. Boyce (VAN)
5	3rd transversality condition	Dr. Boyce (VAN)
6	3rd Jacobi equations	unassigned
7	4th transversality condition	unassigned

The analytic approach will not be detailed. The end conditions are defined by the mission criteria formulation and the transversality conditions. We will assume a formal expansion for the solution of the differential equations; Chapter II of reference 2 gives such an expansion. The differential equations solutions will be substituted into the end conditions. The result will be infinite series in time at cut-off. The coefficients of the expansion will be functions of initial and end conditions. These expansions will be inverted for the initial λ's and time at cut-off. This inversion may be attempted by formal means. Only the University of North Carolina and Marshall Space Flight Center have considered this problem to date. Other contractors may be assigned special parts of this problem.

The impulse optimization by North American Aviation will be continued until this field is thoroughly covered.

There is only one paper on celestial mechanics. It is by Dr. Mary Payne of Republic Aviation Corporation. In this paper the origin of the coordinate system is accelerated and the coordinate system is rotating. The coordinates are the initial values in the solution of Euler's problem of two fixed centers. The origin is selected in such a way as to minimize the effects of non-integrable terms in the perturbation equations. Work will continue on this problem to determine the value of the approach.

There are three papers in the field of large computer exploitation. The first of these is by the Northeast Louisiana State College Department of Mathematics group. This paper presents a recursion process which is to be used in least squares curve fitting. In the process the points taken from the trajectory are designated by a parameter β_i where β_i is the ordered m-tuple,

$[t_{i0}, t_{i1}, \ldots t_{im}]$. There seems no need for this restriction. The vector $[t_{i0}, t_{i1}, \ldots t_{im}]$ may be any m functions such that χ may be considered a function of these functions. This is implied by the last paragraph.

We take as given values t_{ij} (i = 1 ... n, the points; j = 1 ... n (functions)) and χ_i. Define

1) $\overline{g}_j = \overline{t}_j \quad i = 1 \ldots m$

2) $\overline{g}'_\gamma = \overline{g}_\gamma - (\overline{g}_\gamma \cdot \overline{e}_0)\,\overline{e}_0 - (\overline{g}_\gamma \cdot \overline{e}_1)\,\overline{e}_1 - \ldots - (\overline{g}_\gamma \cdot \overline{e}_{\gamma-1})\,\overline{e}_{\gamma-1}$

$$\left\| \overline{g}'_\gamma \right\| = \sqrt{\overline{g}'_\gamma \cdot \overline{g}'_\gamma}$$

$$\overline{e}_\gamma = \frac{\overline{g}'_\gamma}{\left\| \overline{g}'_\gamma \right\|}$$

Let γ successively take on values $\gamma = 0, 1, \ldots, m$. and compute $\overline{g}'_\gamma$, $\left\| \overline{g}'_\gamma \right\|$, and $\overline{e}_\gamma$ where $\overline{g}'_0 = \overline{g}_0$

3) Define $\overline{\chi} = [\chi_1, \ldots, \chi_n]$

4) $A_j = \overline{\chi} \cdot \overline{e}_j \qquad j = 0, \ldots, m.$

Mr. Vance of Chrysler presented a method for solving the above problem in the second progress report. It differs in outlook and procedure from the above.

The present report by Mr. Vance of Chrysler purports to present "one way of solving the problem of data point selection for the generation of a least squares approximation of a multivariate function by a linear combination of polynomials which are orthonormal over a region." It is needless to say this problem needs our attention. The work developes only the concepts. Much added work is required before the method is usable. Time has not permitted the search of references. As a result certain questions remain. Some of these are: (1) what are the details of the derivations of equations on page 197,(2) what modifications are needed to replace w_i with density considerations? Mr. Vance will continue to work in this area. The paper shows potential for success in developing a means of selecting points for least squares curve fitting.

The last paper of this report is by Shigemichi Suzuki and Sylvia M. Hubbard of the University of North Carolina. Linear programming techniques are used to fit known functions. The linear programming techniques should better control the errors at injection. The work presented is to be implemented at Marshall Space Flight Center in the near future.

There are several unanswered questions in the multivariant functional model development. Four of these questions are listed.

1) What terms should be selected for the polynomial?

2) What criteria should be used in developing the polynomial?

3) What points should be selected for fitting the polynomial?

4) What is the best functional form for fitting the data?

Suzuki and Hubbard's contribution will let us start to look at questions two and four. Vance's work attacks the third question. No theoretical work has been done on the first question.

The accomplishments in this field have been large. The least squares approach has been used. Polynomials have been used as the functional form. This gives an engineering answer to questions two and four. Empirical methods have been used to answer questions one and three. The results are sufficient for the needs of today.

A complete steering angle program and a time-to-fly program have been developed here in the past for a specific vehicle. The time required for the development of each step is recorded below.

a. One week to gather data if data is readily available.

b. Three days to establish the performance for the 1st stage.

c. One day to obtain the volume of 1st stage trajectories.

d. One day to obtain the volume of second stage trajectories and invert the matrix. ($\frac{1}{2}$ hr for calculus of variations plus $\frac{1}{2}$ hr for transferring data plus one hour for matrix inversion for a total of 2 hrs machine time).

e. One day to compute coefficients (15 minutes of machine time).

f. Two days to record and check coefficients.

g. One day to check selected χ curve by simulation runs ($1\frac{1}{2}$ hour machine time).

The above were actual times required. The large time for small computer runs was caused by data handling and delays between computer runs. It will be noted that two days were lost in recording data. Two days were also lost by human error. Automation of the process is underway. There is no known way to reduce the data gathering time. Complete automation should make the remainder of the problem an over-night job. It will also cut the cost of human error. Automation of the process will make procedures static. It is costly and it should be done only when the state of the art permits. Whether the state of the art permits such automation now is an open question.

REFERENCES

1. Hunt, Robert W., "Utilization of the Accessory Minimum Problem in Trajectory Analysis," MTP-AERO-62-74, October 2, 1962.

2. Moulton, F. R., "Differential Equations," Dover Publishing, Inc., 1930.

DEPARTMENT OF MATHEMATICS

VANDERBILT UNIVERSITY

AN APPLICATION OF CALCULUS OF VARIATIONS TO THE OPTIMIZATION OF MULTISTAGE TRAJECTORIES

By

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AN APPLICATION OF CALCULUS OF VARIATIONS TO THE OPTIMIZATION OF MULTISTAGE TRAJECTORIES

By

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16302

SUMMARY

A procedure is developed for determining a fuel minimizing trajectory for a multistage rocket in three dimensional space. In each stage the fuel burning rate and magnitude of thrust are assumed constant. The motion is subject to the inverse square gravity law but with negligible atmospheric resistance. The Euler-Lagrange equations determine minimizing trajectories in a given stage. Transversality conditions are then invoked to extend a minimizing path across the boundary to the next stage. The existence of minimizing trajectories is assumed, sufficient conditions not being investigated in this paper.

In Appendix I a simple form of Zermelo's navigation problem, extended to several stages, is solved to illustrate some aspects of multistage problems.

Appendix II gives a summary of necessary conditions for calculus of variations problems of the Mayer form involving control variables.

I. INTRODUCTION

The problem is to determine the fuel minimizing trajectory of a rocket whose flight consists of several stages caused by engine shutoffs at specified times. Initial position and velocity are assumed given and target conditions specified. In each stage the analytic formulation is similar to that of Cox and Shaw (Ref. 1), and we make their basic assumptions that the earth can be considered spherical, the inverse square gravity law holds, the only forces acting on the rocket are thrust and gravity, the direction of thrust is the axial direction of the rocket, rotation effects can be ignored, in each stage the magnitude of thrust and the fuel burning rate are constant, and the center of mass of the rocket is fixed with respect to the rocket.

The general procedure is roughly as follows. Using the fixed initial conditions for the first stage, determine as solutions of the Euler-Lagrange equations the family of minimizing trajectories satisfying those conditions. The given time t_1 for the end of the first stage will fix on each minimizing trajectory a definite point. The totality of these points will constitute a subspace S_1 , which will be the locus of initial points for the second stage. New values of mass, thrust, and fuel burning rate determine new Euler-Lagrange equations. Minimizing trajectories must satisfy these new equations in this stage and also must satisfy transversality conditions for initial points in subspace S_1 . Through each point of S_1 these conditions determine a unique trajectory, and on each of these trajectories the given time t_2 for the end of the second stage will fix a definite point. The totality of these points will be subspace S_2 , which in turn will be the locus of initial points for the third stage, and transversality conditions will again determine a family of minimizing trajectories, one issuing from each point of S_2 . This procedure is repeated until in the final stage the mission objectives will impose criteria for selecting a pieced trajectory satisfying the given initial conditions and extending through the several stages. Closed form solutions are not attainable in most cases. However, it would seem possible to extend the single stage adaptive guidance mode computational procedures through several successive stages.

II. FORMULATION OF THE PROBLEM

A plumbline coordinate system is used (Ref. 1, p.108; Ref. 2, p.11), with the center of mass of the rocket designated by $\underline{x} = (x_1,x_2,x_3)$ and its velocity by $\underline{u} = (u_1,u_2,u_3)$. The time t is taken as independent variable, and $\underline{u} = d\underline{x}/dt.$. The thrust vector $\underline{F} = (0,F,0)$, having its magnitude F constant for each stage, is assumed to be directed along the axis of the rocket. The orientation of the rocket axis relative to the plumbline system is designated by $\underline{\chi} = (\chi_1,\chi_2,\chi_3)$, where χ_1,χ_2,χ_3 are the pitch, roll, and yaw angles, respectively.

If $\underline{\ddot{x}}_g$ denotes the gravitational acceleration and $[A]$ the matrix for transformation of vectors from the missle to the plumbline coordinate system, then Newton's second law gives as equations of motion of the rocket

$$\underline{\dot{u}} = m^{-1}\, \underline{F}\, [A] + \underline{\ddot{x}}_g \ , \quad \underline{\dot{x}} = \underline{u} \ . \tag{1}$$

In terms of pitch, roll, and yaw, the matrix A has the following form (Ref. 1, p.108; Ref. 2, p.26):

$$A = \begin{bmatrix} CPCR & SPCR & SR \\ -SPCY - CPSRSY & CPCY - SPSRSY & CRSY \\ SPSY - CPSRCY & -CPSY - SPSRCY & CRCY \end{bmatrix} \qquad \begin{matrix} CP = \cos\chi_1 \\ SP = \sin\chi_1 \\ \text{etc.} \end{matrix}$$

Since roll effects are to be ignored, the roll χ_2 will be assumed identically zero. Hence $CR = 1$, $SR = 0$, and the variable χ_2 may be dropped. Since fuel consumption is monotonically increasing with time, minimization of time of flight is equivalent to minimizing fuel consumption. It is more convenient to treat the problem from the minimum time standpoint.

In the terminology of the general theory of Appendix II we now have state variables u_1, u_2, u_3, x_1, x_2, x_3, control variables χ_1 and χ_3, and independent variable t. The function to be minimized, the function $h(\underline{b})$ in the appendix, is simply the final time t_f. Hence t_f is one of the parameters in $\underline{b}$; other parameters may occur in the initial and end conditions and in stage boundary conditions. The mass m is assumed a known function of t in each stage so is not included in the state variables.

Thus the problem is to find in a class of admissible sets of functions $\underline{u}(t)$, $\underline{x}(t)$, $\underline{\chi}(t)$ and parameters $\underline{b}$ a set that will satisfy the differential equations (1) and the given end conditions and that will minimize the final time t_f.

III. FIRST STAGE

Let the time interval for the first stage be $t_o \leqq t \leqq t_1$, and the initial conditions, $\underline{u}(t_o) = \underline{u}_o$, $\underline{x}(t_o) = \underline{x}_o$. On putting $\chi_2 \equiv 0$ in A and using $\mu r^{-3}\underline{x}$ for $\underline{\ddot{x}}_g$, where μ is the gravitational constant times the mass of the earth, we get equations (1) in the form

$$
\begin{aligned}
\dot{u}_1 &= -Fm^{-1}SPCY - \mu r^{-3}x_1 \\
\dot{u}_2 &= Fm^{-1}CPCY - \mu r^{-3}x_2 \\
\dot{u}_3 &= Fm^{-1}SY - \mu r^{-3}x_3 \\
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= u_3
\end{aligned}
\tag{2}
$$

In order to apply the necessary conditions of Appendix II, we now define a generalized Hamiltonian

$$
\begin{aligned}
H = {} & L_1(-Fm^{-1}SPCY - \mu r^{-3}x_1) + L_2(Fm^{-1}CPCY - \mu r^{-3}x_2) \\
& + L_3(Fm^{-1}SY - \mu r^{-3}x_3) + L_4u_1 + L_5u_2 + L_6u_3.
\end{aligned}
$$

By condition I, Appendix II, the Euler-Lagrange equations are

$$
\dot{u} = H_{L_i}, \quad \dot{x}_i = H_{L_{i+3}}, \quad \dot{L}_i = -H_{u_i}, \quad \dot{L}_{i+3} = -H_{x_i}, \quad H_{\chi_j} = 0, \quad i = 1,2,3, \quad j = 1,3.
$$

These formulas give the six equations (2) plus the following eight:

$$\begin{aligned}
\dot{L}_1 &= -L_4 \\
\dot{L}_2 &= -L_5 \\
\dot{L}_3 &= -L_6 \\
\dot{L}_4 &= \mu r^{-3}L_1 - 3\mu r^{-5}x_1(L_1x_1 + L_2x_2 + L_3x_3) \\
\dot{L}_5 &= \mu r^{-3}L_2 - 3\mu r^{-5}x_2(L_1x_1 + L_2x_2 + L_3x_3) \\
\dot{L}_6 &= \mu r^{-3}L_3 - 3\mu r^{-5}x_3(L_1x_1 + L_2x_2 + L_3x_3)
\end{aligned} \tag{3}$$

$$\begin{aligned}
&L_1 CPCY + L_2 SPCY = 0 \\
&L_1 SPSY - L_2 CPSY + L_3 CY = 0
\end{aligned} \tag{4}$$

Assuming $CY \neq 0$ and letting $D^2 = L_1^2 + L_2^2$, $E^2 = L_1^2 + L_2^2 + L_3^2$, $D > 0,\ E > 0$

we get from equations (4) that

$$\begin{aligned}
&\tan \chi_1 = -L_1/L_2 \ , \ SP = -L_1/D \ , \ CP = L_2/D \ , \\
&\tan \chi_3 = L_3/D \ , \ SY = L_3/E \ , \ CY = D/E \ ,
\end{aligned} \tag{5}$$

the choice of signs in SP,CP,SY,CY being a consequence of the Weierstrass and Clebsch conditions, as will be shown in the next section. From (5) it follows that the thrust vector in the plumbline system can be expressed as

$$\underline{F}\,[A] = F(-SPCY, CPCY, SY) = F(L_1/E, L_2/E, L_3/E) \ .$$

Equations (5) may be used to eliminate the control variables from equations (2), thus giving, together with equations (3), a system of 12 differential equations of the first order in 12 dependent variables. This system may be written as six equations of second order, which in vector notation are

$$\begin{aligned}
\ddot{\underline{x}} &= F\underline{E}/mE - \mu\underline{x}/r^3, \\
\ddot{\underline{E}} &= -\mu\underline{E}/r^3 + 3\mu(\underline{x}\cdot\underline{E})\underline{E}/r^5,
\end{aligned} \tag{6}$$

where $\underline{E}$ denotes the vector (L_1,L_2,L_3).

Although the result is not utilized in this paper, it is of interest to note that three first integrals of the system (6) can be readily obtained by the following device. Cross multiply the first of equations (6) by $\underline{E}$ and the second by $\underline{x}$ and add the resulting equations to get

$$\underline{E} \times \ddot{\underline{x}} + \underline{x} \times \ddot{\underline{E}} = 0 . \tag{7}$$

This now yields

$$\underline{E} \times \dot{\underline{x}} + x \times \dot{\underline{E}} = \underline{M} , \tag{8}$$

where $\underline{M}$ is a constant vector, since the derivative with respect to t of the left member of (8) is the left member of (7).

The equations (2) and (3), after elimination of the control variables, or, equivalently, system (6), will have a six-parameter family of solutions satisfying the given initial conditions $\underline{u}(t_o) = \underline{u}_o, \underline{x}(t_o) = \underline{x}_o$. However, since the equations are homogeneous in the L's, if $\underline{u}(t), \underline{x}(t), \underline{L}(t)$ is a solution, then so is $\underline{u}(t), \underline{x}(t), c\underline{L}(t)$ for any non-zero constant c . Thus, if initial values of the L's are taken as parameters, only their ratios are significant in determining $\underline{u}(t), \underline{x}(t)$. Hence the value of one L may be fixed, or some function of the L's may be assigned a value at $t = t_o$, say $L_1^2(t_o) + L_2^2(t_o) + L_3^2(t_o) = 1$. Thus there is a five-parameter family of trajectories satisfying the Euler-Lagrange equations and having the given initial values. If $b_1, \dots, b_5$ denote the parameters, the equations of the family may be written

$$\begin{aligned} \underline{u} &= \underline{u}(t, b_1, b_2, b_3, b_4, b_5) , \\ \underline{x} &= \underline{x}(t, b_1, b_2, b_3, b_4, b_5) . \end{aligned} \tag{9}$$

Each of these curves is the path of least time from the initial point to any other point on it, assuming that a minimum exists and that only one of the curves joins the two points. (The geometrical terminology refers to the seven dimensional space $t, \underline{u}, \underline{x}$ and not to three dimensional physical space.) Putting $t = t_1$ gives a point on each curve, and the totality of such points constitutes a subspace S_1 . If S_1 is considered as a given locus of variable end-points for the first stage, then, since t has constant value t_1 on S_1 , each trajectory is a time minimizing trajectory from the initial point to S_1 , and hence must satisfy the transversality conditions at S_1 . This property will be utilized in Section VI.

IV. THE WEIERSTRASS AND CLEBSCH CONDITIONS

We now show that, with the choice of signs adopted in (5), the necessary conditions II and III of Appendix II are satisfied by solutions of equations (2), (3), (4). For the Weierstrass test let circumflexes denote arbitrary values of the control variables. Then

$$H(t,\underline{u},\underline{x},\underline{\chi},\underline{L}) - H(t,u,x,\hat{\chi},L)$$

$$= Fm^{-1}(-L_1SPCY + L_2CPCY + L_3SY + L_1S\hat{P}C\hat{Y} - L_2C\hat{P}C\hat{Y} - L_3S\hat{Y})$$

$$= Fm^{-1}(E + L_1S\hat{P}C\hat{Y} - L_2C\hat{P}C\hat{Y} - L_3S\hat{Y}) > 0 ,$$

as is implied by the general inequality

$$(a^2 + b^2 + c^2)^{1/2} \geqq (a \sin A + b \cos A) \cos B + c \sin B ,$$

which holds for all real values of a, b, c, A, B.

For the Clebsch test, the matrix of the quadratic form involved is

$$\begin{bmatrix} L_1SPCY - L_2CPCY & L_1CPSY + L_2SPSY \\ L_1CPSY + L_2SPSY & L_1SPCY - L_2CPCY - L_3SY \end{bmatrix} .$$

By virtue of equations (5) this becomes

$$\begin{bmatrix} -D^2/E & 0 \\ 0 & -E \end{bmatrix} ,$$

which implies that the quadratic form is negative definite.

There are in all four sets of values of SP, CP, SY, CY in terms of the L's that will satisfy equations (4). Two of them reverse the inequality signs in conditions II and III, but there is one other set besides that given in (5) that satisfies conditions II and III. It can be got from (5) by replacing D by $-D$. This amounts to changing χ_1 to $\chi_1 + \pi$ and χ_3 to $\pi - \chi_3$, and it is found that this actually produces the same direction of thrust as before.

V. SECOND AND SUBSEQUENT STAGES

For the second stage the range of t is $t_1 \leqq t \leqq t_2$. The initial point is required to be in S_1, the equations of which are obtained by putting $t = t_1$ in (9):

$$\left.\begin{aligned} \underline{u} &= \underline{u}(t_1,b_1,b_2,b_3,b_4,b_5) \\ \underline{x} &= \underline{x}(t_1,b_1,b_2,b_3,b_4,b_5) \end{aligned}\right\} \equiv \underline{X}_1(\underline{b}) , \qquad (10)$$

the six functions in the right members being denoted by $\underline{X}_1(\underline{b})$ to conform with the notation in Appendix II. The function $T_1(\underline{b})$ is the constant t_1.

The differential equations of motion are of the same form as for the first stage, although F and m have different values. To allow for possible discontinuities in the L's, we denote their right hand limits at t_1 by $L(t_1+)$. There are five transversality conditions (Condition I, Appendix II) which must be satisfied at $t = t_1$:

$$\underline{L}(t_1+) \cdot \underline{X}_{1b_k} = 0 , \qquad k = 1,2,3,4,5. \qquad (11)$$

Since these equations are homogeneous in the L's, and so are the equations analogous to (2), (3), and (4), it follows that for the determination of $\underline{u}(t)$ and $\underline{x}(t)$ again only the ratios of the L's are significant. Thus again there will be an eleven parameter family of minimizing trajectories. When values are given to the b's to fix a point in S_1, there will be six values $\underline{u}(t_1)$, $\underline{x}(t_1)$ and five transversality conditions to determine the eleven parameters. This in general will fix a unique minimizing trajectory issuing from each point of S_1. Let the equations of these trajectories be expressed by the same equations (9) as for the first stage except that now the range for t is from t_1 to t_2. Putting $t = t_2$ will determine a definite point on each trajectory, and the locus of these points will be a subspace S_2 with equations

$$\left.\begin{array}{l} \underline{u} = \underline{u}(t_2, b_1, b_2, b_3, b_4, b_5) \\ \underline{x} = \underline{x}(t_2, b_1, b_2, b_3, b_4, b_5) \end{array}\right\} \equiv \underline{X}_2(\underline{b}) .$$

Note that again the transversality and other conditions involving the end point need not be used to determine the five parameter family of trajectories but only the conditions at the initial point.

For subsequent stages the procedure is like that for the second stage. The initial point for the third stage would be restricted to subspace S_2 and transversality conditions involving $\underline{X}_2(\underline{b})$ and $\underline{L}(t_2+)$ would be used.

The computational procedure given by Cox and Shaw (Ref. 1, pp 118) could be used in the first stage. Modifications would be needed in the other stages to approximate the partial derivatives of the $\underline{X}(\underline{b})$ functions and to solve the transversality equations.

In the final stage the mission objectives must be fulfilled at the end point. Since there is little hope for closed form solutions, the proposed procedure is to estimate initial conditions and use them to extend a solution by approximate integration methods through the several stages. If the objectives are not attained, make new estimates of initial conditions and new computations of a minimizing trajectory, continuing thus until a trajectory is obtained that achieves the desired objectives with sufficient accuracy.

VI. CONTINUITY PROPERTIES OF THE LAGRANGE MULTIPLIERS

In each stage the trajectories which are without corners and which satisfy the Euler-Lagrange equations will have Lagrange multipliers that are continuous and differentiable (Ref. 3, pp 202-204; Ref. 6, pp 12). However, on passing from one stage to the next, there are discontinuities in the functions defining $\dot{\underline{u}}$. From equations (2) it follows that there will be corners for the functions $\underline{u}$, and hence discontinuities might be expected in the L's . But the functions defining $\dot{\underline{x}}$ and $\dot{\underline{L}}$ are continuous in t, $\underline{u}$, $\underline{x}$ and have continuous partial derivatives. Thus continuous solutions for the L's can be obtained by taking $\underline{u}$ continuous across boundaries, provided the transversality conditions can be satisfied.

In obtaining the family of solutions of the Euler-Lagrange equations in each stage the homogeneity of the equations in the L's was utilized to decrease the number of parameters by one, say by assigning an initial value to one of the L's . As remarked at the end of Section III, the five transversality conditions for parameters $b_1, \ldots, b_5$, namely,

$$\underline{L}(t_1-) \cdot \underline{X}_{1b_k}(\underline{b}) = 0 , \qquad k = 1,2,3,4,5,$$

are satisfied on S_1 . These conditions are the same as conditions (11) in $\underline{L}(t\ +)$ which hold for S_1 as locus of initial points in stage two. Hence $L_1(t_1-), \ldots, L_6(t_1-)$ and $L_1(t_1+), \ldots, L_6(t_1+)$ are proportional. By assigning equal values to one pair from the two sets, all can be made continuous at t_1 .

The transversality condition involving the final time as parameter in each stage is not homogeneous in the L's because of the term h_{b_k} This condition would make the set of L's unique and not necessarily continuous across the boundary; however, it is not essential to use this condition for the determination of the trajectory equations. Hence it is possible to obtain Lagrange multipliers that are continuous through the several stages and to use their ratios at the initial point $t = t_o$ as parameters $b_1, \ldots, b_5$ for a five parameter family extending through all the stages.

APPENDIX I

A MULTISTAGE NAVIGATION PROBLEM

A simple form of Zermelo's navigation problem (Ref. 4), extended to multiple stages, serves to illustrate some features of trajectory problems. Zermelo stated his problem for air flight in a plane, but we follow Cicala's formulation (Ref. 5, pp 19) and consider a motor boat on a plane water surface. A rectangular coordinate system is associated with the plane surface, and the boat is considered a point (x,y). The water current is assumed to have known velocity components u and v as functions of x and y and the time t. Let the velocity vector of the boat relative to the water make an angle θ with the positive x-axis and assume that the magnitude of the velocity vector is a known constant in each stage. The path of the boat is determined by the control variable θ, and the problem is to find θ as a function of t so as to minimize the time t_f for the boat to go from the origin to a specified point (x_f,y_f) that is assumed remote enough to require three stages. In order to get a problem that will have an easily obtained closed form solution, we take the water velocity components to be constants and choose the coordinate system so that $u = 0$, $v = a$.

The problem then is to find functions $x(t)$, $y(t)$, $\theta(t)$ such that

$$\dot{x} = v \cos \theta , \quad \dot{y} = a + v \sin \theta ; \tag{1}$$

$$v = v_1 \text{ for } 0 \leqq t < t_1; \; v = v_2 \text{ for } t_1 \leqq t < t_2; \; v = v_3 \text{ for } t_2 \leqq t;$$

$$x(0) = y(0) = 0 ; \quad x(t_f) = x_f , \; y(t_f) = y_f ;$$

and such that t_f is a minimum.

First Stage.

As in Appendix II, define the generalized Hamiltonian

$$H = L_1 v_1 \cos \theta + L_2(a + v_1 \sin \theta) .$$

From this H the Euler-Lagrange equations are found to be

$$\dot{x} = v_1 \cos\theta , \quad \dot{y} = a + v_1 \sin\theta ,$$
$$\dot{L}_1 = 0 , \quad \dot{L}_2 = 0 , \quad -L_1 \sin\theta + L_2 \cos\theta = 0 . \tag{2}$$

Hence L_1 and L_2 are constants, say $L_1 = L_{11}$, $L_2 = L_{21}$. It then follows that θ is constant, and integration of the first two of the above equations gives

$$x = (v_1 \cos\theta)t , \quad y = (a + v_1 \sin\theta)t , \tag{3}$$

on using initial conditions $x = y = 0$ when $t = 0$. Thus paths of minimum time are straight lines.

If our problem were a one-stage problem with end point (x_1,y_1) and time t_1 to be a minimum, we would have for the determination of θ, t_1, L_{11} and L_{21} the following equations

$$x_1 = (v_1 \cos\theta)t_1 , \quad y_1 = (a + v_1 \sin\theta)t_1 , \tag{4}$$

$$-L_{11} \sin\theta + L_{21} \cos\theta = 0 , \tag{5}$$

plus the transversality condition

$$L_{11}v_1 \cos\theta + L_{21}(a + v_1 \sin\theta) = 1 . \tag{6}$$

Equation (6) is found from the transversality equation in Appendix II by putting

$k = 1$, $b_1 = t_1$, $T_1 = 0$, $T_2 = t_1$, $X_{11} = 0$, $X_{21} = 0$, $X_{12} = x_1$, $X_{22} = y_1$, $h = t_1$.

Equations (4) determine θ and t_1 , while (5) and (6) give unique multipliers

$$L_{11} = \cos\theta/(v_1 + a \sin\theta), \quad L_{21} = \sin\theta/(v_1 + a \sin\theta) . \tag{7}$$

Now if we consider (x_1,y_1) variable and inquire as to the locus of such points each of which is reached in a minimum time equal to t_1 , we get from (4) with θ variable that the locus of (x_1,y_1) is the circle with center $(0,at_1)$ and radius v_1t_1 .

Second Stage.

The locus of initial points for the second stage is the circle mentioned in the preceding sentence. We write it as

$$x_1 = (v_1 \cos \alpha)t_1, \quad y_1 = (a + v_1 \sin \alpha)t_1 \tag{8}$$

with the parameter α replacing the θ of equations (4) since we shall continue to use θ as the control variable. The differential equations of constraint for this stage are the same as for the first stage except that v_2 replaces v_1.

The Euler-Lagrange equations are as before, with v_2 replacing v_1, and hence L_1 and L_2 are constant, say $L_1 = L_{12}$, $L_2 = L_{22}$. It follows that θ is constant.

If the end point for the second stage is considered fixed at (x_2, y_2), then transversality conditions for parameters α and t_2 are

$$\begin{aligned} &L_{12}v_1t_1 \sin \alpha - L_{22}v_1t_1 \cos \alpha = 0 , \\ &L_{12}v_2 \cos \theta + L_{22}(a + v_2 \sin \theta) = 1 . \end{aligned} \tag{9}$$

The first of these equations, together with the last of the Euler-Lagrange equations, implies that $\theta = \alpha$. Then, from the pair of equations (9), it follows that

$$L_{12} = \cos \theta/(v_2 + a \sin \theta), \quad L_{22} = \sin \theta/(v_2 + a \sin \theta) . \tag{10}$$

Thus L_{12} and L_{22} are not equal to L_{11} and L_{21}, indicating discontinuities in the multipliers at stage boundaries. However, the control variable θ is continuous, being in fact the same constant in the two stages.

On integrating the Euler-Lagrange equations for x and y and using (8) as initial conditions, one finds that

$$\begin{aligned} x &= (v_2 \cos \theta)t + (v_1 - v_2)t_1 \cos \theta , \\ y &= (a + v_2 \sin \theta)t + (v_1 - v_2)t_1 \sin \theta . \end{aligned} \tag{11}$$

For each constant θ , the path is a straight line.

Now consider the locus of end points (x_2,y_2) that will each be reached in minimum time t_2 . Fixing $t = t_2$ in (11) and considering θ variable shows the locus to be the circle with center $(0,at_2)$ and radius $v_1t_1 + v_2(t_2 - t_1)$.

Third Stage.

With the circle of the preceding sentence as locus of initial points, the end point is required to be (x_f,y_f) and time t_f is to be a minimum. In the same way as before the path is shown to be a straight line with the control variable constant and equal to its value in the preceding stages. The new equations for x and y are

$$\begin{aligned} x &= (v_3 \cos \theta)t + \left[(v_1 - v_2)t_1 + (v_2 - v_3)t_2 \right] \cos \theta , \\ y &= (a + v_3 \sin \theta)t + \left[(v_1 - v_2)t_1 + (v_2 - v_3)t_2 \right] \sin \theta . \end{aligned} \tag{12}$$

By putting the given values x_f,y_f in equations (12), one can solve for the minimum time $t = t_f$ and for the constant control angle θ . Then equations (11) with $t = t_2$, $x = x_2$, $y = y_2$ and equations (8) determine the corner points (x_1,y_1) and (x_2,y_2) .

Conclusions.

This problem illustrates the extension of a trajectory across stage boundaries where the differential equations of constraint are discontinuous. The effect of the homogeneity in the Lagrange multipliers is similar to that in the more general problem.

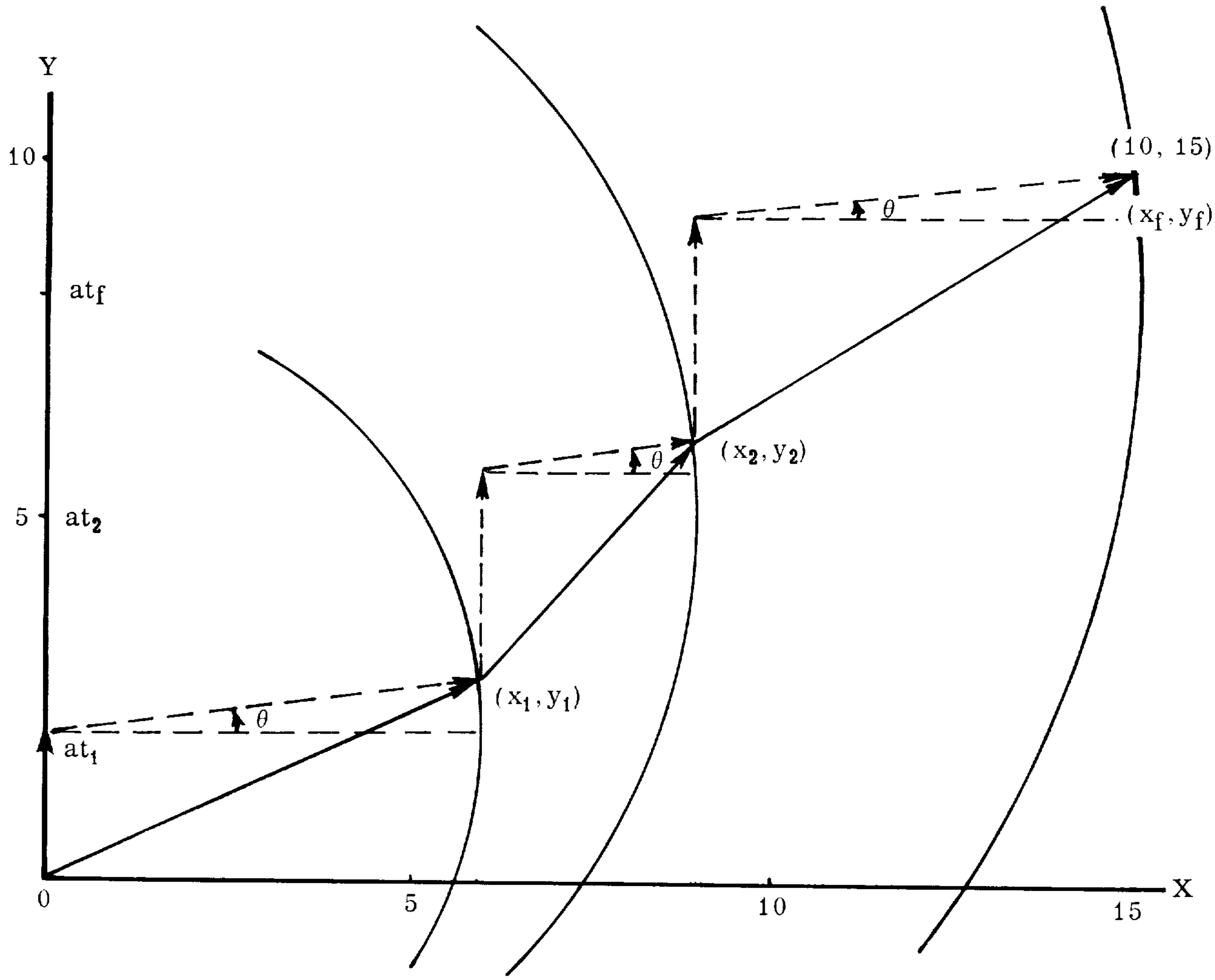
The unique Lagrange multipliers that satisfy the Euler-Lagrange equations and the transversality conditions of I, Appendix II, are discontinuous at stage boundaries. However, the ratio $L_2/L_1 = \tan \theta$ is the same for each stage. The equations containing L's are homogeneous in the L's , except that the transversality condition computed for the final time as parameter in each stage is not homogeneous. But this transversality condition is not needed to determine the family of minimizing trajectories which satisfy initial conditions in each stage. That is, in order to obtain a pieced trajectory extending through the several stages, only the ratio of the L's is needed, and, since the ratio is preserved, the L's may be chosen continuous.

The geometrical interpretation of this problem is of interest, so a diagram is shown below. The computations were made for

$$a = 1,\ t_1 = 2,\ t_2 = 5,\ v_1 = 3,\ v_2 = 1,\ v_3 = z,\ x_f = 15,\ y_f = 10,$$

and yielded the results

$$t_f = 8.09,\ \theta = 7^\circ 14',\ (x_1,y_1) = (5.93,\ 2.76),\ (x_2,y_2) = (8.89,\ 6.13).$$



APPENDIX II

NECESSARY CONDITIONS FOR MAYER PROBLEMS WHICH CONTAIN CONTROL VARIABLES

This appendix lists the principal classical necessary conditions, modified to allow for control (or undifferentiated) variables in the constraints. For proofs and fuller discussion see References 6, 7, 8.

Notation

$\underline{x} = (x_1, \ldots, x_n)$	state variables, functions of independent variable t
$\underline{y} = (y_1, \ldots, y_m)$	control variables, functions of t
$\underline{b} = (b_1, \ldots, b_r)$	parameters occurring in end conditions
$T_1, \underline{X}_1 = (X_{11}, \ldots, X_{1n})$	functions of $\underline{b}$ defining first end point
$T_2, \underline{X}_2 = (X_{21}, \ldots, X_{2n})$	functions of $\underline{b}$ defining second end point
$\underline{g} = (g_1, \ldots, g_n)$	functions of (t,x,y), defining derivative constraints
$\underline{L} = (L_1, \ldots, L_n)$	Lagrange multipliers, functions of t
$H = \underline{L} \cdot \underline{g}$	generalized Hamiltonian function
$h(\underline{b})$	function to be minimized

Variables occurring as subscripts will denote partial derivatives, and a superimposed dot will indicate differentiation with respect to t. A set $t, \underline{x}, \underline{y}, \underline{b}$ will be called admissible if it belongs to a given open set R, and a set $\underline{x}(t), \underline{y}(t), \underline{b}$ will be an admissible arc if its elements are all admissible and if $\underline{x}(t)$ is continuous and $\dot{\underline{x}}(t), \underline{y}(t)$ are piecewise continuous. The functions occurring in T, $\underline{X}$, $\underline{g}$, and h are assumed to have continuous partial derivatives of at least the second order.

Statement of Problem

In a given class of admissible functions and parameters $\underline{x}(t),\underline{y}(t),\underline{b}$ it is required to find a set which satisfies the differential equations and end conditions

$$\dot{\underline{x}} = \underline{g}(t,\underline{x},\underline{y}) \ , \ t_1 \leqq t \leqq t_2$$

$$t_1 = T_1(\underline{b}) \ , \ t_2 = T_2(\underline{b}) \ , \ \underline{x}(t_1) = \underline{X}_1(\underline{b}) \ , \ \underline{x}\ (t_2) = \underline{X}_2(\underline{b})$$

and which minimizes the given function $h(\underline{b})$.

Let C be an admissible arc $\underline{x}(t)$, $\underline{y}(t)$, $\underline{b}$ which is a solution of the problem. Also let C be assumed normal (Ref. 6, pp 15) and to have $\dot{\underline{x}}(t)$ and $\dot{\underline{y}}(t)$ continuous. Then C must satisfy the following four conditions.

I. First Necessary Condition. For every minimizing arc C there exist unique multipliers $L_i(t)$, having continuous first derivatives, such that the equations (Euler-Lagrange)

$$\dot{x}_i = H_{L_i} \ , \ \dot{L}_i = -H_{x_i} \ , \ H_{y_j} = 0, \quad i = 1,\ldots,n, \quad j = 1,\ldots,m,$$

hold along C . Also the end values of C satisfy the transversality conditions

$$H_1 T_{1b_k} - \underline{L}_1 \cdot \underline{X}_{1b_k} - H_2 T_{2b_k} + \underline{L}_2 \cdot \underline{X}_{2b_k} + h_{b_k} = 0, \quad k = 1,\ldots,r,$$

where subscripts 1 and 2 on H and $\underline{L}$ indicate evaluation for $t = t_1$ and $t = t_2$, respectively.

As a consequence of the above Euler-Lagrange equations it follows that also along a minimizing arc C

$$dH/dt = H_t \ ,$$

and hence that, if H does not involve t explicitly, then H is constant along C .

II. Weierstrass Condition. Along a minimizing arc C the inequality

$$H(t,\underline{x},\underline{Y},\underline{L}) \leqq H(t,\underline{x},\underline{y},\underline{L})$$

must hold for every admissible element $(t,\underline{x},\underline{Y})$.

III. Clebsch (Legendre) Condition. At each element $(t,\underline{x},\underline{y},\underline{L})$ of a minimizing arc C the inequality

$$\sum_{i,j=1}^{m} H_{y_i y_j} Y_i Y_j \leqq 0$$

must hold for every set $(Y_1,\ldots,Y_m)$.

IV. Second Order Condition. The second variation of h along a minimizing arc C is non-negative for every variation of C satisfying the equations of variation.
(Cf. Ref. 6, pp 16.) No use of this condition is made in this paper.

REFERENCES

1. J. G. Cox and W. A. Shaw, "Preliminary Investigations on Three Dimensional Optimum Trajectories," Progress Report No. 1 on Studies in the Fields of Space Flight Guidance and Theory, pp 103-123, NASA - MSFC, MTP - AERO - 61 - 91, Dec. 18, 1961.

2. W. E. Miner, "Methods for Trajectory Computation," NASA - MSFC, Aeroballistics Internal Note No. 3-61, May 10, 1961.

3. G. A. Bliss, "Lectures on the Calculus of Variations," The University of Chicago Press, Chicago, 1946.

4. E. Zermelo, "Ueber die Navigation in der Luft als Problem der Variationsrechnung," Jahresbericht der Deutschen Mathematiker - Vereinigung, Angelegenheiten, 39 (1930), pp 44-48.

5. P. Cicala, "An Engineering Approach to the Calculus of Variations," Libreria Editrice Universitaria Levrotto and Bella, Torino, Italy, 1957.

6. M. R. Hestenes, "A General Problem in the Calculus of Variations with Applications to Paths of Least Time," The Rand Corporation Research Memorandum RM-100, Santa Monica, California, March, 1950.

7. C. R. Cavoti, "The Calculus of Variations Approach to Control Optimization," Progress Report No. 2 on Studies in the Fields of Space Flight and Guidance Theory, pp 75-168, NASA - MSFC, MTP - AERO - 62 - 52, June 26, 1962.

8. L. D. Berkovitz, "Variational Methods in Problems of Control and Programming, "Journal of Mathematical Analysis and Applications, 3 (1961), pp 145-169.

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TWO-POINT BOUNDARY-VALUE PROBLEM OF THE CALCULUS OF VARIATIONS FOR OPTIMUM ORBITS

By

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DEFINITION OF SYMBOLS

$\underline{R}$	Vehicle position vector
r	Distance to vehicle
$\underline{V}$	Velocity vector of vehicle
v	Speed of vehicle
$\underline{\xi}$	Perturbation displacement vector
$f, g, \dot{f}, \dot{g}$	Coordinate functions
μ	Mass parameter
t	Time
t_F	Time at which the natural end point is reached
k	Magnitude of thrust
$\underline{T}$	Direction of thrust
m	Mass of vehicle
λ, γ, σ	Lagrange multipliers or adjoint variables
a	Semi major axis
n	Mean motion
d_i	$\underline{R}_i \cdot \dot{\underline{R}}_i$
θ	Incremental eccentric anomaly
f_1, f_2, f_3, f_4	Functions of θ defined by Eqs. (48)
$\{\lambda\}$	Adjoint variables defined by Eq. (18)
$\{r\}$	State variables defined by Eq. (18)
$\{\delta(t_F)\}$	Residual vector defined by Eq. (19)
$\{\alpha\}$	Variational parameters
$\{p\}, \{q\}$	Defined by Eqs. (21), (22), (23)
$[\Phi]$	Partial derivatives of state variables as defined by Eq. (25)

$[\Lambda]$	Partial derivatives of adjoint variables as defined by Eq. (26)
$[F], [G], [J]$	Defined by Eqs. (27), (28), (29)

Subscripts

u	Unperturbed solution
o	Value at the initial time t_o
E	Value at the natural end point
A, B	Values corresponding to variational parameter set A or B

Superscripts

k	Value at the kth iteration

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TWO-POINT BOUNDARY-VALUE PROBLEM OF THE CALCULUS OF VARIATION FOR OPTIMUM ORBITS

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SUMMARY

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This report contains a description for the solution of the two-point boundary-value problem of the calculus of variations for optimum orbits.

The method employed uses Lagrange multipliers and Pontryagin's maximum principle to obtain the decision functions.

In addition, two differential correction schemes are described. The first scheme is a "method by forward integration," and the second is an alternate "method by backward integration" that attempts to reduce the difficulties that might be encountered in inverting a differential correction matrix.

The optimum orbit is determined by a perturbation method similar to that of Encke and accommodates hyperbolic as well as elliptic orbits. The equations necessary for the generation of a digital-computer program are derived.

INTRODUCTION

The usual methods of solving the two-point boundary-value problem of the calculus of variations involve the use of iterative gradient techniques. With these methods, the desired solution is reached only after making a great number of incremental variations and examining the changes that these variations cause. As one might expect, the rate of convergence for this method is very slow.

Another method of solving the two point boundary value problem of the calculus of variations, which will be described in this report, is one where all the decision functions and trajectories that are being used are extremals. This method uses, in addition to the state variables Lagrange multipliers or adjoint variables that play the key role in deciding the optimal direction of thrust, time of thrust duration, etc. The adjoint variables also define the natural end-point conditions by which the two-point boundary-value problem can be terminated. This natural end point, in general, will not be the desired end point. A differential correction scheme provides the means of obtaining another optimum trajectory the natural end point of which will be closer to the desired end point.

EQUATIONS OF MOTION

In a Newtonian system, the equations of motion of a particle that is in the gravitational field of N attracting bodies and is subject to other accelerations, such as thrust, drag, oblateness, radiation pressure, etc., are given by

$$\ddot{\underline{R}}_V = -\sum_{K=1}^{N} \mu_{B_K} \frac{\underline{R}_{VB_K}}{r^3_{VB_K}} + \sum_j \underline{F}_j \tag{1}$$

The problem that will be considered here is one in which the vehicle is in the gravitational field of only one body and is subjected to a variable thrust $\underline{k}$. In this case, Eq. (1) is reduced to

$$\ddot{\underline{R}} = -\mu \frac{\underline{R}}{r^3} + \frac{k}{m} \underline{T} \tag{2}$$

where $\underline{T}$ is a unit vector in the direction of thrust. The magnitude of the thrust is taken to be proportional to the mass flow and is given by

$$k = -c\dot{m} \tag{3}$$

The constant of proportionality c is related to the more commonly used constant specific impulse I_{sp} by

$$c = I_{sp} g \tag{4}$$

DERIVATION OF OPTIMIZATION EQUATIONS

In the derivation of the optimization equations, it will be assumed that the vehicle can have two possible values of thrust, either $k = k_{max}$ or $k = k_{min}$. The magnitudes of these two thrust values may differ with each stage.

Minimum-Fuel Condition

The value of the integral to be minimized is given by

$$I = \int_{t_o}^{t_F} -dm = \int_{t_o}^{t_F} -\dot{m}\, dt \tag{5}$$

and the conditions of constraint are given by

$$\dot{\underline{V}} + \frac{\mu \underline{R}}{r^3} - \frac{k}{m} \underline{T} = 0$$

$$\dot{\underline{R}} - \underline{V} = 0 \tag{6}$$

$$\dot{m} + \frac{k}{c} = 0$$

Because these conditions of constraint are satisfied at every point on the trajectory, we may rewrite Eq. (5), without changing its value, as

$$\begin{aligned} I &= \int_{t_o}^{t_F} \left[-\dot{m} + \underline{\lambda} \cdot (\dot{\underline{V}} + \frac{\mu \underline{R}}{r^3} - \frac{k}{m} \underline{T}) + \underline{\gamma} \cdot (\dot{\underline{R}} - \underline{V}) + \sigma (\dot{m} + \frac{k}{c}) \right] dt \\ &= \int_{t_o}^{t_F} L\,(\dot{\underline{R}}, \underline{R}, \dot{\underline{V}}, \underline{V}, \dot{m}, m, \underline{\lambda}, \underline{\gamma}, \sigma)\, dt \end{aligned} \tag{7}$$

where $\underline{\lambda}(t)$, $\underline{\gamma}(t)$, and $\sigma(t)$ are undetermined Lagrange multipliers that are chosen so as to determine the optimum decision functions required to solve the problem.

Applying the Euler Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{8}$$

to the state variables, results in the following set of equations:

$$\dot{\underline{\lambda}} + \underline{\gamma} = 0$$

$$\dot{\underline{\gamma}} - \frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} \underline{R} = 0 \tag{9}$$

$$\dot{\sigma} - \frac{k}{m^2} \underline{\lambda} \cdot \underline{T} = 0$$

Equations (6) and (9) can be combined to form

$$\ddot{\underline{R}} = -\frac{\mu \underline{R}}{r^3} + \frac{k}{m}\underline{T}$$

$$\ddot{\underline{\lambda}} = -\frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^5}\underline{R} \qquad (10)$$

$$k = -c\dot{m}$$

$$\dot{\sigma} = \frac{k}{m^2}\underline{\lambda}\cdot\underline{T}$$

In addition, the natural boundary conditions are

$$\underline{\lambda}\cdot\delta\underline{V}\Big|_{t_o}^{t_F} = 0$$

$$\underline{\gamma}\cdot\delta\underline{R}\Big|_{t_o}^{t_F} = 0 \qquad (11)$$

$$(\sigma - 1)\delta m\Big|_{t_o}^{t_F} = 0$$

Because variations in the position and velocity at the end points are zero, the first two expressions of Eq. (11) yield no additional information about the values of $\underline{\lambda}$ and $\underline{\gamma}$ at the end points. The variation of mass at the final end point, however, is not zero, i.e., $\delta m(t_F) \neq 0$. Hence, the only way to satisfy the third expression of Eq. (11) is to demand that

$$\sigma(t_F) - 1 = 0 \qquad (12)$$

The only additional information that is necessary to completely define the extremal is the determination of the optimum thrust vector and the duration of this thrust.

For the determination of this decision function, we make use of Pontryagin's "Maximum Principle," [1,2] which states that a necessary condition for an integral of the form of Eq. (7) to be minimized is that the Hamiltonian be a maximum. The Hamiltonian for this problem is given by

$$H = \underline{\lambda} \cdot \dot{\underline{V}} + \underline{\gamma} \cdot \underline{V} + \sigma \dot{m}$$

$$= \underline{\lambda} \cdot \left[-\frac{\mu}{r^3} \underline{R} + \frac{k}{m} \underline{T} \right] - \dot{\underline{\lambda}} \cdot \dot{\underline{R}} - \frac{\sigma k}{c} \tag{13}$$

$$= \left[-\frac{\mu \underline{R} \cdot \underline{\lambda}}{r^3} - \dot{\underline{\lambda}} \cdot \dot{\underline{R}} \right] + k \left[\frac{\underline{\lambda} \cdot \underline{T}}{m} - \frac{\sigma}{c} \right]$$

For H to be a maximum, the unit thrust vector $\underline{T}$ must be in the direction of $\underline{\lambda}$, or

$$\underline{T} = \frac{\underline{\lambda}}{|\lambda|} \tag{14}$$

Therefore, the coefficient of k in Eq. (13), which is defined as the switch function, becomes

$$S = \frac{|\lambda|}{m} - \frac{\sigma}{c} \tag{15}$$

The necessary conditions that must be placed on the magnitude of the thrust for H to be a maximum are the following:

$$\begin{aligned} &\text{if } S > 0 \quad \text{then} \quad k = k_{max} \\ &\text{if } S < 0 \quad \text{then} \quad k = k_{min} \end{aligned} \tag{15a}$$

Furthermore, when thrust is applied, it is desirable to make the switch function as large as possible. This can be accomplished by allowing the mass to be as small as permissible, which implies the obvious condition that any empty tanks or other unnecessary weight be dropped as soon as possible.

Minimum-Time Condition

In this case, the value of the integral to be minimized is given by

$$I = \int_{t_o}^{t_F} dt = \int_{t_o}^{t_F} \left[1 + \underline{\lambda} \cdot \left(\dot{\underline{V}} + \frac{\mu \underline{R}}{r^3} - \frac{k}{m} \underline{T} \right) + \underline{\gamma} \cdot (\dot{\underline{R}} - \underline{V}) + \sigma \left(\dot{m} + \frac{k}{c} \right) \right] dt \tag{16}$$

Application of the Euler Lagrange equations and Pontryagin's Principle lead to the exact same results as the minimum-fuel condition, with the exception of one of the natural-boundary conditions. In place of the third expression in Eq. (11), we now have

$$\sigma\,\delta m \Big|_{t_o}^{t_F} = 0$$

or

$$\sigma\,(t_F) = 0 \tag{17}$$

Therefore, for the "minimum-time" condition the natural end point occurs when $\sigma = 0$.

ITERATION SCHEME

General Procedure

The problem is to generate a set of initial adjoint variables such that an optimum orbit can be computed where the natural end point matches the desired end point. (The end points are, of course, given by terminal values of the state variables.) With initial values of the state variables specified and an estimate for the initial values of the adjoint variables, an iterative method can be used to solve this problem. Improved estimates for the initial values of the adjoint variables can be obtained by computing the residuals or differences between the values of the state variables at the desired end point and the natural end point and then applying a differential correction matrix to these residuals. We define the $\{r\}$, $\{\lambda\}$, and $\{\delta\,(t_F)\}$ vectors as

$$\{r\} = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ m \end{Bmatrix}, \qquad \{\lambda\} = \begin{Bmatrix} -\dot{\lambda}_x \\ -\dot{\lambda}_y \\ -\dot{\lambda}_z \\ \lambda_x \\ \lambda_y \\ \lambda_z \\ \sigma \end{Bmatrix}, \tag{18}$$

and

$$\{\delta(t_F)\} = \begin{Bmatrix} x\,(t_F) - x_E \\ y\,(t_F) - y_E \\ z\,(t_F) - z_E \\ \dot{x}\,(t_F) - \dot{x}_E \\ \dot{y}\,(t_F) - \dot{y}_E \\ \dot{z}\,(t_F) - \dot{z}_E \\ m(t_F) - m_E \end{Bmatrix} \tag{19}$$

where the subscript E denotes the values of the state variables at the desired end point.

The Kth approximation to $\{\lambda(t_o)\}$ is designated by $\{\lambda^{(K)}(t_o)\}$, and it is desired to obtain an improved value of $\{\lambda^{(K+1)}(t_F)\}$. The procedure is as follows: using $\{\lambda^{(K)}(t_o)\}$ in the integration scheme, the position, velocity and mass at time t_F, as well as the residuals $\{\delta^{(K)}(t_F)\}$, are computed; and the initial values of the adjoint variables are then changed so as to reduce the residuals,

$$\left\{\lambda^{(K+1)}(t_o)\right\} = \left\{\lambda^{(K)}(t_o)\right\} + \left\{\Delta\lambda^{(K)}(t_o)\right\} \tag{20}$$

where $\{\Delta\lambda^{(K)}(t_o)\}$ is to be found by using a differential correction matrix.

Methods for Obtaining the Differential Correction Matrix

Making use of Eqs. (14) and (18), the first two expressions of Eq. (10) can be written as follows:

$$\begin{aligned} \{\dot{r}\} &= \left\{q(\{r\},\{\lambda\})\right\} \qquad \text{or} \qquad \dot{r}_i = q_i(\{r\},\{\lambda\}) \\ \{\dot{\lambda}\} &= \left\{p(\{r\},\{\lambda\})\right\} \qquad \text{or} \qquad \dot{\lambda}_i = p_i(\{r\},\{\lambda\}) \end{aligned} \tag{21}$$

where

$$\begin{aligned} q_1 &= \dot{x} \\ q_2 &= \dot{y} \\ q_3 &= \dot{z} \\ q_4 &= -\frac{\mu x}{r^3} + \frac{k}{m}\frac{\lambda_x}{|\lambda|} \\ q_5 &= -\frac{\mu y}{r^3} + \frac{k}{m}\frac{\lambda_y}{|\lambda|} \\ q_6 &= -\frac{\mu z}{r^3} + \frac{k}{m}\frac{\lambda_z}{|\lambda|} \\ q_7 &= -\frac{k}{c} \end{aligned} \tag{22}$$

and

$$p_1 = \frac{\mu \lambda_x}{r^3} - \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} x$$

$$p_2 = \frac{\mu \lambda_y}{r^3} - \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} y$$

$$p_3 = \frac{\mu \lambda_z}{r^3} - \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} z$$

$$p_4 = \dot{\lambda}_x \tag{23}$$

$$p_5 = \dot{\lambda}_x$$

$$p_6 = \dot{\lambda}_z$$

$$p_7 = \frac{k}{m^2} \; |\lambda|$$

Taking the variations of Eq. (21) with respect to a set of parameters $\{\alpha\} = (\alpha_1, \alpha_2, \ldots\ldots \alpha_7)$, we find that

$$\frac{d}{dt} [\Phi] = [F][\Phi] + [G][\Lambda]$$

$$\frac{d}{dt} [\Lambda] = -[F]^* [\Lambda] + [J][\Phi] \tag{24}$$

where

$$[\Phi] = \frac{\partial\{r\}}{\partial\{\alpha\}} \doteq \begin{bmatrix} \frac{\partial x(t)}{\partial \alpha_1} & \frac{\partial x(t)}{\partial \alpha_2} & \frac{\partial x(t)}{\partial \alpha_3} & \frac{\partial x(t)}{\partial \alpha_4} & \frac{\partial x(t)}{\partial \alpha_5} & \frac{\partial x(t)}{\partial \alpha_6} & \frac{\partial x(t)}{\partial \alpha_7} \\ \frac{\partial y(t)}{\partial \alpha_1} & & & & & & \\ \frac{\partial z(t)}{\partial \alpha_1} & & & & & & \\ \frac{\partial \dot{x}(t)}{\partial \alpha_1} & & & & & & \\ \frac{\partial \dot{y}(t)}{\partial \alpha_1} & & & & & & \\ \frac{\partial \dot{z}(t)}{\partial \alpha_1} & & & & & & \\ \frac{\partial m(t)}{\partial \alpha_1} & & & & & & \frac{\partial m(t)}{\partial \alpha_7} \end{bmatrix} \tag{25}$$

$$[\Lambda] = \frac{\partial\{\lambda\}}{\partial\{\alpha\}} \equiv \begin{bmatrix} \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_1} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_2} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_3} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_4} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_5} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_6} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_7} \\ \frac{-\partial\dot{\lambda}_y(t)}{\partial\alpha_1} & & & & & & \\ \frac{-\partial\dot{\lambda}_y(t)}{\partial\alpha_1} & & & & & & \\ \frac{\partial\lambda_x(t)}{\partial\alpha_1} & & & & & & \\ \frac{\partial\lambda_y(t)}{\partial\alpha_1} & & & & & & \\ \frac{\partial\lambda_z(t)}{\partial\alpha_1} & & & & & & \\ \frac{\partial\sigma(t)}{\partial\alpha_1} & & & & & & \frac{\partial\sigma(t)}{\partial\alpha_7} \end{bmatrix} \qquad (26)$$

$$
[F] = \frac{\partial\{q\}}{\partial\{r\}} \equiv \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{-\mu}{r^3} + \frac{3\mu x^2}{r^5} & \frac{3\mu xy}{r^5} & \frac{3\mu xz}{r^5} & 0 & 0 & 0 & \frac{-k\lambda_x}{m^2\lambda} \\
\frac{3\mu xy}{r^5} & \frac{-\mu}{r^3} + \frac{3\mu y^2}{r^5} & \frac{3\mu yz}{r^5} & 0 & 0 & 0 & \frac{-k\lambda_y}{m^2\lambda} \\
\frac{3\mu xz}{r^5} & \frac{3\mu yz}{r^5} & \frac{-\mu}{r^3} + \frac{3\mu z^2}{r^5} & 0 & 0 & 0 & \frac{-k\lambda_z}{m^2\lambda} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \qquad (27)
$$

$$
[G] = \frac{\partial\{q\}}{\partial\{\lambda\}} \equiv \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{k}{m\lambda} - \frac{k\lambda_x^2}{m\lambda^3} & \frac{-k\lambda_x\lambda_y}{m\lambda^3} & \frac{-k\lambda_x\lambda_z}{m\lambda^3} & 0 \\
0 & 0 & 0 & \frac{-k\lambda_x\lambda_y}{m\lambda^3} & \frac{k}{m\lambda} - \frac{k\lambda_y^2}{m\lambda^3} & \frac{-k\lambda_y\lambda_z}{m\lambda^3} & 0 \\
0 & 0 & 0 & \frac{-k\lambda_x\lambda_z}{m\lambda^3} & \frac{-k\lambda_y\lambda_z}{m\lambda^3} & \frac{k}{m\lambda} - \frac{k\lambda_z^2}{m\lambda^3} & 0
\end{bmatrix} \qquad (28)
$$

$$[J] = \frac{\partial\{p\}}{\partial\{r\}} \equiv -\begin{bmatrix} 6\mu\lambda_x x + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^5}(1 - \frac{5x^2}{r^2}) & \frac{3\mu}{r^5}(\lambda_x y + x\lambda_y) - \frac{15\mu}{r^7}xy(\underline{R}\cdot\underline{\lambda}) & \frac{3\mu}{r^5}(\lambda_x z + x\lambda_z) - \frac{15\mu}{r^7}xz(\underline{R}\cdot\underline{\lambda}) & 0 & 0 & 0 \\ \frac{3\mu}{r^5}(\lambda_y x + y\lambda_x) - \frac{15\mu}{r^7}xy(\underline{R}\cdot\underline{\lambda}) & \frac{6\mu\lambda_y y}{r^5} + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^5}(1 - \frac{5y^2}{r^2}) & \frac{3\mu}{r^5}(\lambda_y z + y\lambda_z) - \frac{15\mu}{r^7}yz(\underline{R}\cdot\underline{\lambda}) & 0 & 0 & 0 \\ \frac{3\mu}{r^5}(\lambda_z x + z\lambda_x) - \frac{15\mu}{r^7}xz(\underline{R}\cdot\lambda) & \frac{3\mu}{r^5}(\lambda_z y + z\lambda_y) - \frac{15\mu}{r^7}yz(\underline{R}\cdot\underline{\lambda}) & \frac{6\mu\lambda_z z}{r^5} + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^3}(1 - \frac{5z^2}{r^5}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

<u>Method of Forward Integration</u>. Two convenient sets of parameters to work with are the sets that consist of the <u>initial</u> values of the state variables and adjoint variables, which are, respectively

$$\{\alpha\}_A = \left\{ r\,(t_o) \right\}$$

and

$$\{\alpha\}_B = \left\{ \lambda\,(t_o) \right\} \tag{29}$$

Using these sets of parameters, Eq. (24) can be integrated "forward" simultaneously with equations of motion, using the initial values of $[\Phi]$ and $[\Lambda]$ as given by

$$\left\{ \begin{matrix} \left[\Phi_A(t_o) \right] = I \\ \\ \left[\Lambda_A(t_o) \right] = 0 \end{matrix} \right\} \text{ and } \left\{ \begin{matrix} \left[\Phi_B(t_o) \right] = 0 \\ \\ \left[\Lambda_B(t_o) \right] = I \end{matrix} \right\} \tag{30}$$

The differential corrections are obtained by solving the system of equations

$$\begin{aligned} \left\{ \Delta r\,(t_F) \right\} &= \left[\Phi_A(t_F) \right] \left\{ \Delta r\,(t_o) \right\} + \left[\Phi_B(t_F) \right] \left\{ \Delta \lambda\,(t_o) \right\} \\ \left\{ \Delta \lambda\,(t_F) \right\} &= \left[\Lambda_A(t_F) \right] \left\{ \Delta r\,(t_o) \right\} + \left[\Lambda_B(t_F) \right] \left\{ \Delta \lambda\,(t_o) \right\} \end{aligned} \tag{31}$$

and, because

$$\left\{ \Delta r\,(t_o) \right\} = 0 \quad \text{and} \quad \left\{ \Delta r\,(t_F) \right\} = \left\{ \delta\,(t_F) \right\}$$

we find

$$\left\{ \Delta \lambda\,(t_o) \right\} = \left[\Phi_B(t_F) \right]^{-1} \left\{ \delta\,(t_F) \right\} \tag{32}$$

An interesting feature of this differential correction scheme is a tendency for the inverse of the differential correction matrix $[\Phi_B(t_F)]$ to become more and more singular as the time arc increases. This tendency toward singularity is a problem of utmost interest.

<u>Method of Backward Integration</u>. If the use of double-precision techniques fails to provide the required numerical accuracy for the inverse of the matrix, an alternate method of generating the differential correction matrix can be used. This alternate scheme employs a method of "backward" integration to provide a differential correction matrix consisting of the sum of two matrices, only one of which requires inversion to produce the differential corrections. In this case, the two sets of parameters consist of the <u>final</u>

values of the state variables and adjoint variables, which are, respectively

$$\{\alpha\}_A = \{ r(t_F) \}$$

and

$$\{\alpha\}_B = \{ \lambda (t_F) \} \tag{33}$$

Using these sets of parameters, the variational Eq. (24) can be integrated "backward." The procedure is as follows: the equations of motion are integrated "forward" until the natural end point is reached; the residuals are computed; and, then, Eq. (24), together with the equations of motion, are simultaneously integrated "backward" starting at time t_F and ending at time t_o, using for initial values of $[\Phi]$ and $[\Lambda]$:

$$\left\{ \begin{array}{l} \left[\Phi_A(t_F)\right] = I \\ \left[\Lambda_A(t_F)\right] = 0 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} \left[\Phi_B(t_F)\right] = 0 \\ \left[\Lambda_B(t_F)\right] = I \end{array} \right\} \tag{34}$$

The differential corrections are obtained by solving the equations

$$\begin{aligned} \{\Delta\lambda(t_o)\} &= \left[\Lambda_A(t_o)\right]\{\Delta r(t_F)\} + \left[\Lambda_B(t_o)\right]\{\Delta\lambda(t_F)\} \\ \{\Delta r(t_o)\} &= \left[\Phi_A(t_o)\right]\{\Delta r(t_F)\} + \left[\Phi_B(t_o)\right]\{\Delta\lambda(t_F)\} \end{aligned} \tag{35}$$

and, because, in this case,

$$\{\Delta r(t_o)\} = 0 \quad \text{and} \quad \{\Delta r(t_F)\} = \{\delta(t_F)\}$$

solving Eq. (35) for $\{\Delta\lambda(t_o)\}$, we find that

$$\{\Delta\lambda(t_o)\} = \left[[\Lambda_A(t_o)] - [\Lambda_B(t_o)][\Phi_B(t_o)]^{-1}[\Phi_A(t_o)]\right]\{\delta(t_F)\} \tag{36}$$

Convergence of Iteration

Several difficulties are connected with the above iteration scheme, and some of them might be crucial enough to cause divergence of the iteration. These difficulties might arise for the following reasons:

1. In the variational equations, the variation of burning time is not accounted for.

2. The inversion of a matrix is required in both methods to obtain the differential-correction matrix. Furthermore this inversion becomes more involved since the residual $m(t_F)-m_E$ of the vector $\{\delta(t_F)\}$ is unspecified and requires additional computation.

3. The change Δt_F in the final time has not been taken into account. However, this should be included by considering the additional transversality condition which results in $\{\lambda\} \cdot \{r\} = 0$.

DIGITAL PROGRAM

Trajectory Equations

The equations that completely define the trajectory have been described previously. The order in which these equations are programmed for the general case (with thrust) is as follows:

$$s = \left(\frac{|\lambda|}{m} - \frac{\sigma}{c}\right) \qquad \begin{matrix} >0 & k = k_{max} \\ <0 & k = k_{min} \end{matrix}$$

$$\dot{m} = -\frac{k}{c}$$

$$\dot{\sigma} = \frac{k\,|\lambda|}{m^2}$$

$$\frac{d}{dt}[\Phi] = [F][\Phi] + [G][\Lambda]$$

$$\frac{d}{dt}[\Lambda] = -[F]^*[\Lambda] + [J][\Phi] \tag{37}$$

$$\ddot{\underline{R}} = -\frac{\mu \underline{R}}{r^3} + \frac{k}{m}\,\frac{\underline{\lambda}}{|\lambda|}$$

$$\ddot{\underline{\lambda}} = -\frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu(\underline{\lambda} \cdot \underline{R})}{r^5}\,\underline{R}$$

$$m = m(t) + \int_t^{t+\Delta t} \dot{m}\, dt$$

$$\sigma = \sigma(t) + \int_t^{t+\Delta t} \dot{\sigma}\, dt$$

These equations are integrated until the natural end point is reached. At that time, the residuals are computed and compared to a predetermined set of maximum permissible values $\{\epsilon\}$.

If $\delta_j(t_F) \le \epsilon_j$ for all the residuals, then that trajectory is the solution to the two-point boundary-value problem. If $\delta_j(t_F) > \epsilon_j$ for any of the residuals, then a differential correction is applied to the initial values of the adjoint variables as described previously. If the alternate differential correction scheme is used, then a "backward" integration is necessary before any corrections can be applied.

Numerical Procedures

The differential equations of Eq. (37) can be integrated numerically with a Runge-Kutta fourth-order method. To reduce any accumulation of error that might result from a number of step-by-step integration, however, it is convenient to write the equation of motion for the high thrust case in the form

$$\ddot{\underline{R}} = \ddot{\underline{R}}_u + \ddot{\underline{\xi}} \tag{38a}$$

The velocity and position vectors can be written as

$$\begin{aligned} \dot{\underline{R}} &= \dot{\underline{R}}_u + \dot{\underline{\xi}} \\ \underline{R} &= \underline{R}_u + \underline{\xi} \end{aligned} \tag{38b}$$

where $\ddot{\underline{R}}_u$ is the unperturbed solution and $\underline{\xi}$ is the perturbation.

In this method, $\ddot{\underline{R}}_u$ is taken as

$$\underline{R}_u = \frac{k}{m}\,\underline{T}_i = -\frac{c\dot{m}}{m}\,\underline{T}_i \tag{39}$$

and

$$\ddot{\underline{\xi}} = -\frac{\mu \underline{R}}{r^3} + \frac{k}{m}\left[\underline{T} - \underline{T}_i\right] \tag{40}$$

Eq. (40) is integrated numerically, and the solution to Eq. (39) is

$$\begin{aligned} \underline{R}_u &= f\,\underline{R}(t_i) + g\,\dot{\underline{R}}(t_i) + h\underline{T}(t_i) \\ \dot{\underline{R}}_u &= \dot{f}\,\underline{R}(t_i) + \dot{g}\,\dot{\underline{R}}(t_i) + \dot{h}\underline{T}(t_i) \end{aligned} \tag{41}$$

where

$$f = 1$$

$$g = t - t_i$$

$$h = -c\left\{\frac{1}{\dot{m}}\left[m \log m - m_i \log m_i - (m - m_i)\right] - (t - t_i) \log m_i\right\}$$

$$\dot{f} = 0$$

$$\dot{g} = 1$$

$$\dot{h} = -c(\log m - \log m_i)$$

$$m = m_i + (t - t_i)\,\dot{m}$$

(the subscript i refers to values at time t_i)

This perturbation method, or Encke scheme as it is commonly called, will reduce inaccuracies occurring in numerical integration, provided that the perturbation terms are small compared with the total solution. Whenever these perturbations become too large, a rectification takes place, i.e., an initialization occurs in which the values of the variable at time t now becomes the values of the variable at time t_i. A rectification takes place whenever any of the following conditions occur:

$$\left|\frac{\underline{\xi}}{r}\right| > \epsilon_{pos} \qquad \text{(position rectification)}$$

$$\left|\frac{\dot{\underline{\xi}}}{\dot{r}}\right| > \epsilon_{vel} \qquad \text{(velocity rectification)} \tag{42}$$

$$\sqrt{2\,\underline{T} \cdot \underline{T}_i} > \epsilon_{acc} \qquad \text{(acceleration rectification)}$$

SOLUTION OF EQUATIONS FOR THE COASTING STAGES

The solution of the equations of motion and the Euler Lagrange equations can be derived in closed form for the coasting period. In the no thrust region (k=0), the equation of motion reduces to

$$\underline{R} = -\frac{\mu \ddot{\underline{R}}}{r^3} \qquad \text{(Kepler problem)} \tag{43}$$

The two-body orbit that results from the solution of Eq. (43) with the initial conditions

$$\underline{R}(t_i) = \underline{R}_i$$
$$\dot{\underline{R}}(t_i) = \dot{\underline{R}}_i \tag{44}$$

can be written as a linear combination of $\underline{R}_i$ and $\dot{\underline{R}}_i$ as

$$\underline{R} = f\,\underline{R}_i + g\,\dot{\underline{R}}_i$$
$$\dot{\underline{R}} = \dot{f}\,\underline{R}_i + \dot{g}\,\dot{\underline{R}}_i \tag{45}$$

The coefficients f, g, $\dot{f}$, and $\dot{g}$ are obtained as follows: we represent the initial conditions by the set of elements

$$a = \left(\frac{2}{r_i} - \frac{v_i^2}{\mu}\right)^{-1}$$
$$d_i = \underline{R}_i \cdot \dot{\underline{R}}_i \tag{46}$$
$$n = \frac{\mu^{1/2}}{a^{3/2}} \qquad \text{(elliptic)}$$
$$n = \frac{\mu^{1/2}}{(-a)^{3/2}} \qquad \text{(hyperbolic)}$$

This results in the following Kepler's equation

$$n(t - t_i) = \theta - \sin\theta + \frac{r_i}{a}\sin\theta + \frac{d_i}{\sqrt{\mu a}}(1 - \cos\theta) \quad \text{(elliptic)}$$
$$n(t - t_i) = \sinh\theta - \theta - \frac{r_i}{a}\sinh\theta + \frac{d_i}{\sqrt{-\mu a}}(\cosh\theta - 1) \quad \text{(hyperbolic)} \tag{47}$$

where $\theta(t)$ is the incremental eccentric anomaly $E - E_i$; the functions f_1, f_2 f_3, f_4 are defined as

$$
\left.
\begin{array}{ll}
f_1(\theta) = \theta - \sin\theta & \\
f_2(\theta) = 1 - \cos\theta & \\
f_3 = \sin\theta = \theta - f_1(\theta) & \text{(elliptic)} \\
f_4 = \cos\theta = 1 - f_2(\theta) & \\
 & \\
f_1(\theta) = \sinh\theta - \theta & \\
f_2(\theta) = \cosh\theta - 1 & \\
f_3(\theta) = \sinh\theta = \theta + f_1(\theta) & \text{(hyperbolic)} \\
f_4(\theta) = \cosh\theta = 1 + f_2(\theta) &
\end{array}
\right\} \tag{48}
$$

and the solution of the two-body problem for both elliptic and hyperbolic orbits is given by

$$
\begin{aligned}
f &= -\frac{|a|}{r_i} f_2 + 1 \\
g &= -\frac{1}{n} f_1 + (t - t_i) \\
\frac{r}{|a|} &= f_2 + \frac{r_i}{|a|} f_4 + \frac{d_i}{\sqrt{\mu |a|}} f_3 \\
\dot{f} &= -\sqrt{\frac{k}{|a|}} \, \frac{1}{r_i} \, \frac{|a|}{r} f_3 \\
\dot{g} &= -\frac{|a|}{r} f_2 + 1 \\
n(t - t_i) &= f_1 + \frac{r_i}{|a|} f_3 + \frac{d_i}{\sqrt{\mu |a|}} f_2
\end{aligned} \tag{49}
$$

For the non-thrust case, we also can solve for $\{\lambda\}$ in closed form. The following is a derivation leading to this closed-form solution: the differential equation for the adjoint variables are written as

$$
\frac{d}{dt} \{\lambda\} = - [F]^* \{\lambda\} \tag{50}
$$

where $[F]$ is defined by Eq. (27); the variational equation for $[\Phi]$ reduces to

$$\frac{d}{dt}[\Phi] = [F][\Phi] \tag{51}$$

taking the transpose of Eq. (50) and postmultiplying by $[\Phi]$, yields

$$\frac{d}{dt}\{\lambda\}^*[\Phi] = -\{\lambda\}^*[F][\Phi] \tag{52}$$

premultiplying Eq. (51) by $\{\lambda\}^*$, yields

$$\{\lambda\}^*\frac{d}{dt}[\Phi] = \{\lambda\}^*[F][\Phi] \tag{53}$$

comparing Eqs. (52) and (53), we see that

$$\{\lambda\}^*\frac{d}{dt}[\Phi] + \frac{d}{dt}\{\lambda\}^*[\Phi] = 0$$

or

$$\frac{d}{dt}\left[\{\lambda\}^*[\Phi]\right] = 0 \tag{54}$$

Eq. (54) states that $\{\lambda\}^*[\Phi]$ is a constant and, therefore, can be written as

$$\{\lambda(t)\}^*[\Phi(t)] = \left\{\lambda(t_K)\right\}^*\left[\Phi(t_K)\right] \tag{55}$$

where t_K is any fixed time in the no-thrust interval; solving Eq. (55) for $\{\lambda(t)\}$, results in

$$\{\lambda(t)\} = \left[\Phi^*(t)\right]^{-1}\left[\Phi^*(t_K)\right]\left\{\lambda(t_K)\right\} \tag{56}$$

In the case where the set of parameters $\{\alpha\}$ corresponds to a set of the state variables $\{\mathbf{r}\}$, the matrix $[\Phi]$ can be written as

$$\left[\Phi_A(t)\right] = \left[\Phi_A(t - t_K)\right]\left[\Phi_A(t_K)\right] \tag{57}$$

taking the transpose and then the inverse of Eq. (57), leads to

$$\left[\Phi^*(t)\right]^{-1} = \left[\Phi_A^*(t - t_K)\right]^{-1}\left[\Phi_A^*(t_K)\right]^{-1} \tag{58}$$

and combining Eqs. (56) and (58), results in

$$\{\lambda(t)\} = \left[\Phi_A^*(t - t_K)\right]^{-1} \{\lambda(t_K)\} \tag{59}$$

which is the closed form solution of $\{\lambda(t)\}$.

The elements of the $\left[\Phi_A(t - t_K)\right]$ matrix are obtained by differentiating the Kepler orbit elements with respect to $\underline{R}(t_K)$ and $\dot{\underline{R}}(t_K)$. The elements of $\left[\Phi_A(t - t_K)\right]_{pq} = \dfrac{\partial r_p(t)}{\partial r_q(t_K)}$, with p, q=1, ..., 7, are as follows:

$$\begin{aligned}
\frac{\partial x_i(t)}{\partial x_j(t_K)} \equiv \frac{\partial x_i}{\partial x_{oj}} &= f\,\delta_{ij} + \frac{3|a|}{r_o^3} x_{oj}\left[x_i - x_{oi} - (t - t_K)\dot{x}_{oi}\right] \\
&+ \frac{|a| x_{oj}}{r_o^3}(\dot{x}_i - \dot{x}_{oi})\left[-3(t - t_K) + g + \frac{r_o}{|a| n}\left(1 - \frac{r_o}{|a|}\right) f_3\right] \\
&- \frac{|a|}{\mu} f_2(\dot{x}_i - \dot{x}_{oi})\dot{x}_{oj} + f_2\left(\frac{1}{r_o} + \frac{1}{|a|}\right)\frac{a^2}{r_o^3} x_{oi} x_{oj}
\end{aligned} \tag{60}$$

$$\begin{aligned}
\frac{\partial x_i(t)}{\partial \dot{x}_j(t_K)} \equiv \frac{\partial x_i}{\partial \dot{x}_{oj}} &= g\,\delta_{ij} + \frac{3|a|}{\mu}\dot{x}_{oj}\left[x_i - x_{oi} - (t - t_K)\dot{x}_{oi}\right] \\
&+ \frac{|a|\dot{x}_{oj}}{\mu}(\dot{x}_i - \dot{x}_{oi})\left[-3(t - t_K) + g + \frac{r_o}{|a| n} f_3\right] \\
&- \frac{|a|}{\mu} x_{oj}(\dot{x}_i - \dot{x}_{oi}) f_2 + \frac{a^2}{\mu r_o} f_2\, x_{oi}\dot{x}_{oj}
\end{aligned}$$

$$\frac{\partial \dot{x}_i(t)}{\partial x_j(t_K)} \equiv \frac{\partial \dot{x}_i}{\partial x_{oj}} = \dot{f}\,\delta_{ij} - \frac{\mu|a|x_{oj}}{r^3 r_o^3}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right]\left[-3(t - t_K) + g + \frac{r_o}{|a|n}(1 - \frac{r_o}{|a|})f_3\right]$$

$$+ \frac{|a|\dot{x}_{oj}}{r^3}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right]f_2 + \frac{r_o}{\mu}(\dot{x}_i - \dot{x}_{oi})\,\dot{x}_{oj}\dot{f}$$

$$- \frac{|a|}{r_o^2}\dot{f}\left(\frac{1}{r_o} + \frac{1}{|a|}\right)x_{oi}\,x_{oj} + \frac{|a|}{r_o^3}(\dot{x}_i - \dot{x}_{oi})\,x_{oj}\left[\dot{g} + \frac{r_o}{r}(1 - \frac{r_o}{|a|})f_4\right]$$

(60 continued)

$$\frac{\partial \dot{x}_i(t)}{\partial \dot{x}_j(t_K)} \equiv \frac{\partial \dot{x}_i}{\partial \dot{x}_{oj}} = \dot{g}\,\delta_{ij} - \frac{|a|\dot{x}_{oj}}{r^3}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right]\left[-3(t - t_K) + g + \frac{r_o}{|a|n}f_3\right]$$

$$+ \frac{|a|}{r^3}f_2\,x_{oj}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right] + \frac{r_o}{\mu}(\dot{x}_i - \dot{x}_{oi})\,\dot{x}_{oj}\left[\frac{|a|}{r_o}\dot{g} + \frac{|a|}{r}f_4\right]$$

$$+ \frac{r_o}{\mu}(\dot{x}_i - \dot{x}_{oi})\,x_{oi}\,\dot{f} - \frac{|a|}{\mu}\dot{f}\,x_{oi}\,\dot{x}_{oj}$$

where i, j=1, 2, 3 correspond to the x, y and z components and

$$x_o \equiv x(t_K)$$

$$r_o \equiv r(t_K)$$

$$r \equiv r(t)$$

The inverse, $[\Phi_A(t - t_K)]^{-1}$, can be obtained from the above expression by replacing

$t \to -t$	$r_o \to r$
$\theta \to \theta$	$x \to x_o$
$r \to r_o$	$x_o \to x$

This results in

$f_1 \to -f_1$	$f \to \dot{g}$
$f_2 \to f_2$	$g \to -g$
$f_3 \to -f_3$	$\dot{f} \to -\dot{f}$
$f_4 \to f_4$	$\dot{g} \to f$

REFERENCES

1. Pontryagin, L. S. "Maximum Principles in the Theory of Optimum Systems" Parts I, II, III, Vol. 20, Soviet Journal on Automation and Remote Control

2. Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F., The Mathematical Theory of Optimal Processes (Interscience Publishers, Inc., New York, 1962)

3. Pines, S., Wolf, H., and Richman, J., "Orbit Determination and General Purpose Differential Correction Program," Republic Aviation Corporation, Report RAC 696 (ARD 708-450)

RESEARCH DEPARTMENT

GRUMMAN AIRCRAFT ENGINEERING CORPORATION

AN APPLICATION OF A SUCCESSIVE APPROXIMATION SCHEME TO OPTIMIZING VERY LOW-THRUST TRAJECTORIES

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16804

Summary

This report describes an application of a successive approximation scheme for optimization of low-thrust trajection involving many revolutions about a central body. The equations of motion, written in terms of orbital parameters rather than position and velocity components, are analyzed, and a convenient thrust formula is derived. This formula, together with the variation-of-parameters method of trajectory computation, has been programmed for the IBM 7090.

INTRODUCTION

One of the principal difficulties associated with the application of a numerical optimization scheme to very low-thrust trajectories is the large computer storage required when Cartesian coordinates are used. Many points are needed to compute each trajectory, and when several solutions must be stored, the computer capacity can easily be exceeded. In addition, the geometry of very low-thrust trajectories suggests that the optimum thrust will oscillate in a regular fashion with only very small changes per revolution superimposed on this oscillation. We are concerned with these small changes, and it seems likely that they will become obscured over many orbital periods.

A natural solution to the problem of storage is to employ a variation-of-parameters integration routine. Furthermore, when the entire problem is rewritten in terms of parameters, a thrust formula suggests itself which is a function of the parameters and a set of slowly varying control functions. It is anticipated that these control functions will be sensitive to the small cyclic changes in thrust.

The specific problem to which we have directed our attention is that of minimizing transfer time for a two-dimensional very low-thrust trajectory when only the thrust direction is variable. The formula for the resulting thrust direction contains three control functions, and specific examples indicate that these have the desirable properties we are seeking. The following contains the development of the thrust formula and specific equations for the parameters we have chosen.

DISCUSSION

Let p_i be the parameters which describe the trajectory at any time t, and let

$$\frac{dp_i}{dt} = \mu G_i(p_1, \ldots, p_n, t, \psi) \quad , \quad i = 1, \ldots, n \tag{1}$$

be their differential equations where μ is a small quantity, in our case the thrust acceleration, and ψ is the angle of the thrust vector. If we write the Euler-Lagrange equations for minimizing time with ψ as the control function and these differential equations as side conditions, the equations for the Lagrange multipliers, ℓ_i, become

$$\frac{d\ell_i}{dt} = - \mu \sum_{k=1}^{n} \ell_k \frac{\partial G_k}{\partial p_i} \quad , \quad i = 1, \ldots, n \tag{2}$$

$$0 = - \mu \sum_{k=1}^{n} \ell_k \frac{\partial G_k}{\partial \psi} . \tag{3}$$

Because for our problem ψ appears only in linear combinations of sines and cosines in the G_i, the last equation yields a formula for the tangent of ψ in terms of the ℓ_i and the parameters p_i. But the p_i are slowly varying, and if the partials $\partial G_k/\partial p_i$ are not too large, the ℓ_i vary slowly as well, since their time derivatives also contain the small quantity μ as a factor. Therefore, if we adopt the ℓ_i as new control functions using the tangent formula to replace ψ, we will have substituted n slowly varying quantities for one rapidly changing one. Under favorable circumstances, this would mean both increased accuracy and reduced storage space requirements.

In practice, it is not necessary that all the parameters be constants of the zero thrust or unperturbed motion, but only that they be slowly varying in some sense and that $\partial G_k/\partial p_i$ be of moderate size. In applying our analysis to very low-thrust trajectories, we have chosen, for sake of computational ease, to integrate directly for θ, the angle of the vehicle in the plane of motion, rather than to calculate a time parameter such as the time of perigee. An examination of the differential equation for θ shows that it is well-behaved and does not contribute significantly more to the time rate of change of the ℓ_i than the other parameters. In terms of the classical orbital elements our parameters are

$$\begin{aligned} p_1 &= \sqrt{a(1 - e^2)/k} = h \\ p_2 &= e \cos \omega = q \\ p_3 &= e \sin \omega = s \\ p_4 &= \theta \end{aligned} \tag{4}$$

where $k = Gm$, the universal gravitational constant multiplied by the mass of the central body. The differential equations for these parameters in a two-dimensional very low-thrust trajectory are

$$\frac{dh}{dt} = \frac{T}{m}[h^2/(1 + q \cos\theta + s \sin\theta)]\cos\psi$$

$$\frac{dq}{dt} = \frac{T}{m}[\cos\theta + (q + \cos\theta)/(1 + q\cos\theta + s\sin\theta)]h\cos\psi$$

$$+ \frac{T}{m}[\sin\theta]h\sin\psi \tag{5}$$

$$\frac{ds}{dt} = \frac{T}{m}[\sin\theta + (s + \sin\theta)/(1 + q\cos\theta + s\sin\theta)]h\cos\psi$$

$$- \frac{T}{m}[\cos\theta]h\sin\psi$$

$$\frac{d\theta}{dt} = [(1 + q\cos\theta + s\sin\theta)^2/(h^3k)]$$

where T/m is the thrust acceleration, a small quantity. Substituting Eqs. (5) into Eq. (3) yields a thrust direction formula, and it should be noted that because ψ does not appear in $d\theta/dt$ the multiplier ℓ_4 will not appear in this formula. Defining two auxiliary quantities, A and B, by

$$A = \ell_1[h/(1 + q\cos\theta + s\sin\theta)]$$

$$+ \ell_2[\cos\theta + (q + \cos\theta)/(1 + q\cos\theta + s\sin\theta)] \tag{6}$$

$$+ \ell_3[\sin\theta + (s + \sin\theta)/(1 + q\cos\theta + s\sin\theta)]$$

$$B = \ell_2 \sin\theta - \ell_3 \cos\theta ,$$

we have for $\tan \psi$

$$\tan \psi = A/B$$

With proper attention to signs, we can solve this equation for the sine and cosine of ψ and substitute these into Eqs. (5). When this is done, we have a set of differential equations of the following form:

$$\frac{dp_i}{dt} = H_i(p_i, \ldots, p_4, \ell_1, \ldots, \ell_3)$$

with the ℓ_i as new control functions.

A successive approximation scheme employing these equations has been programmed and check runs have been made. The early runs indicate that the estimates of the time rate of change of the ℓ_i are valid and that the program can be applied without exceeding available computer storage to the problems for which it was designed. Subsequent efforts will be aimed at assessing the scheme's merits and at the possibility of refining it.

HAYES INTERNATIONAL CORPORATION

ORBITAL ELEMENT EQUATIONS FOR OPTIMUM LOW THRUST TRAJECTORIES

By

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ORBITAL ELEMENT EQUATIONS FOR OPTIMUM LOW THRUST TRAJECTORIES

By

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16805

SUMMARY

The three dimensional optimum trajectory relations developed by Messrs J. G. Cox and W. A. Shaw in Reference 1, are transformed into a form that appears more amenable to low thrust trajectory calculations. Orbital element coordinates, commonly used in Celestial Mechanics, are employed due to their slow variation in low thrust applications. Combinations of these elements and a generalized eccentric anomaly are utilized in arranging the resulting equations in a form which does not contain circular singularities.

LIST OF SYMBOLS

a	Semi-major axis
C	Abbreviated notation for cosine
E	Generalized eccentric anomaly (see Fig. 3)
E^*	Parameter linking the eccentric anomaly with the generalized eccentric anomaly E (see Fig. 3)
e	Numerical eccentricity
F	Thurst
F_g	Gravitational force
G	Gravitational constant
$\bar{H}$	Specific angular mementum vector
H_1, H_2, H_3	Components of angular momentum in equatorial axis system.
i	Inclination angle of the plane of motion
k	Specific energy constant
ℓ	Semi-latus rectum (orbital parameter)
M	Mass of attracting body
$\bar{M}$	A vector perpendicular to the line-of-nodes vector $\bar{N}$
m	Vehicle mass
$\bar{N}$	A vector along the line of nodes directed toward the ascending node
$n = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}}$	Mean motion
p	$= (q_1{}^2 + q_2{}^2 + q_3{}^2)^{\frac{1}{2}}$, Absolute value of Lagrange multiplier vector $\bar{q}$
$\bar{q}$	Lagrange multiplier vector in equatorial coordinate system

q_1, q_2, q_3		Lagrange multiplier components in equatorial coordinate system
R		Perturbation attraction force
r	=	$(x_{p1}^2 + x_{p2}^2 + x_{p3}^2)^{\frac{1}{2}}$, Absolute value of position vector $\bar{x}_p$
S		Abbreviated notation for sine
t		Time
$\bar{u}$		Position vector in equatorial coordinate system
u_1, u_2, u_3		Equatorial coordinates
$\bar{v}$		Position vector in the plane of motion, x, y, coordinate system
x, y		Coordinates of an axis system in the plane of motion with the x - axis directed toward the ascending node and the y - axis 90 degrees forward
$\bar{x}_p$		Position vector in plumb-line coordinate system
x_{p1}, x_{p2}, X_{p3}		Plumb-line coordinates

Greek Symbols

$\bar{\alpha}$		Notation parameter representing either $\bar{u}$ or $\bar{q}$
$\bar{\beta}$		Notation parameter representing either
ε		Mean longitude at epoch
ε^*	=	$\varepsilon - \Omega$, "Mean argument" at epoch
ζ	=	$(1-e^2)^{\frac{1}{2}} \cos i$
θ		Position angle from ascending node
$\bar{\lambda}$		Lagrange multiplier vector in plumb-line coordinate system
$\lambda_1, \lambda_2, \lambda_3$		Lagrange multiplier components in plumb-line coordinate system

ν		True anomaly from perigee
Ξ		Eccentric anomaly
ξ	=	$e \sin \omega$
σ	=	$e \cos \omega$
Ω		Longitude of the ascending node
ω		Argument of peri-apsis

Subscripts

N	Parameter is evaluated at the time of passage through the ascending node
o	Indicates initial condition
r	Indicates physical parameter (as apposed to Lagrange multiplier parameter)
λ	Lagrange multiplier parameter
1,2	The subscripted parameter pertains to body 1 or 2 respectively.
1,2,3	Orthogonal cartesian coordinates

Other Notations

$(\dot{\ })$	Denotes the first time derivative
$(\ddot{\ })$	Denotes the second time derivative
$(\ ) \times (\ )$	Multiplication
$(\bar{\ })$	Denotes a vector
$(\bar{\ }) \times (\bar{\ })$	Vector product
$(\bar{\ }) \cdot (\bar{\ })$	Scalar product
$\lvert\ \rvert$	Absolute value

INTRODUCTION

The "minimum fuel" trajectory equations were derived in Reference 1 by Messrs. J. G. Cox and W. A. Shaw utilizing the following assumptions:

1. Spherical earth.
2. Inverse gravity law, $F_g = - \frac{GMm}{r^2}$
3. The only forces acting on the vehicle are thrust and gravity.
4. Rotation effects on the rocket are ignored.
5. Constant fuel burning rate.
6. The center of mass of the vehicle is fixed with respect to the vehicle.

The equations derived under these conditions are:

$$\begin{aligned} \ddot{\bar{x}}_p + \left(\frac{GM}{|\bar{x}_p|^3}\right) \bar{x}_p &= \frac{F}{m\,|\bar{\lambda}|}\,\bar{\lambda} \\ \ddot{\bar{\lambda}} + \left(\frac{GM}{|\bar{x}_p|^3}\right) \bar{\lambda} &= \frac{3GM}{|\bar{x}_p|^5}\left(\bar{\lambda} \cdot \bar{x}_p\right) \bar{x}_p \end{aligned} \qquad (1)$$

where $\bar{x}_p$ is the position vector in the plumbline coordinate system described in Reference 2 and $\bar{\lambda}$ is the corresponding Lagrange multiplier. The computational method of Reference 1 is that of approximate integration starting with initial conditions on $\bar{x}_p$, $\dot{\bar{x}}_p$, $\bar{\lambda}$ and $\dot{\bar{\lambda}}$.

The computational scheme of Reference 1 does not readily lend itself to rapid trajectory calculations for situations in which the thrust force is small in comparison with the vehicle mass. This is due to the large coordinate variations required for relatively small displacements of the vehicle away from the attracting body. Consequently, expressing the equations in a coordinate system whose natural properties are better suited to approximate integration would be advantageous. This study is then a exploratory investigation into the possibilities afforded by the elliptical element coordinate systems used in Celestial Mechanics.

COMPARISON WITH THREE BODY EQUATIONS

The three-body perturbation equations (Reference 3 or 4) may be obtained in the form

$$\begin{aligned} \ddot{\bar{x}}_1 + k^2 (M + m_1) \frac{\bar{x}_1}{|\bar{x}_1|^3} &= \frac{dR_{1,2}}{d\,\bar{x}_1} \\ \ddot{\bar{x}}_2 + k^2 (M + m_2) \frac{\bar{x}_2}{|\bar{x}_2|^3} &= \frac{dR_{2,1}}{d\,\bar{x}_2} \end{aligned} \qquad (2)$$

where $\bar{x}_1$ and $\bar{x}_2$ are the position vectors of the bodies of mass m_1 and m_2 respectively and the $R_{i,j}$ $(i = 1,2;\ j = 1,2)$ are the perturbation attractions of body i on body j.

Equations 2 are usually referred to equatorial or ecliptic axis systems and it is convenient to transform Equations 1 into an equatorial system for the purpose of comparison. The transform relations (Reference 2) are:

$$\begin{aligned} \bar{u} &= \left[\phi_o\right]_1 \left[A_o - 90^o\right]_2 \bar{x}_p \\ \bar{q} &= \left[\phi_o\right]_1 \left[A_o - 90^o\right]_2 \bar{\lambda} \end{aligned} \qquad (3)$$

where $\bar{u}$ is the position vector and $\bar{q}$ is the Lagrange multiplier in the equatorial system. The "minimum fuel" trajectory equations in the equatorial system then become

$$\begin{aligned} \ddot{\bar{u}} + \left(\frac{GM}{r^3}\right) \bar{u} &= \frac{F}{mp} \bar{q} \\ \ddot{\bar{q}} + \left(\frac{GM}{r^3}\right) \bar{q} &= \frac{3GM}{r^5} (\bar{u} \cdot \bar{q})\bar{u} \end{aligned} \qquad (4)$$

where $p = |\bar{q}|$ and $r = |\bar{u}|$. Adding $\frac{GM}{p^3}\bar{q}$ to both sides of the second of Equations 4 yields

$$\begin{aligned} \ddot{\bar{u}} + \left(\frac{GM}{r^3}\right) \bar{u} &= \frac{F}{mp} \bar{q} \\ \ddot{\bar{q}} + \left(\frac{GM}{p^3}\right) \bar{q} &= -\,GM \left[\left(\frac{1}{r^3} - \frac{1}{p^3}\right) \bar{q} - \frac{3}{r^5}(\bar{u}\cdot\bar{q})\bar{u}\right] \end{aligned} \qquad (5)$$

A comparison of Equations 5 with Equations 2, neglecting m_1 and m_2 with respect to M, indicates they are of the same form with $\bar{x}_1 = {}^2\bar{u}$, $\bar{x}_2 = \bar{q}$, $k^2M = GM$, $\partial R_{1,2}/\partial \bar{x}_1 = \partial R_r/\partial \bar{u} = \bar{q}\ F/mp$, and $\partial R_{2,1}/\partial \bar{x}_2 = \partial R_\lambda/\partial \bar{q} = -GM\left[(1/r^3 - 1/p^3)\ \bar{q} - 3/r^5\ (\bar{u}\cdot\bar{q})\ \bar{u}\right]$.

The classical variation of parameter procedure, described in References 3 and 4, may thus be adapted to the trajectory equations, Equations 5.

ORBITAL ELEMENT EQUATIONS

The application of the variation of parameter method to Equations (2) is given in detail in References 3 and 4. The procedure results in two sets of six first order equations in six orbital parameters. The form chosen here has a - the semi-major axis, e - the eccentricity, ε - the mean longitude at epoch, i - the inclination of the orbit plane, ω - the argument of perihelion and Ω - the longitude of the ascending node as variables. Only one set of equations is presented; however, it must be remembered that this set actually represents two sets of the same form, one representing the real or physical situation coming from the first of Equations 5, and the second representing the Lagrange multiplier equations coming from the second of Equations 5.

The equations are

$$\dot{a} = \frac{2}{na}\ \frac{\partial R}{\partial \varepsilon}$$

$$\dot{e} = -\frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\left\{\left[1-(1-e^2)^{\frac{1}{2}}\right]\frac{\partial R}{\partial \varepsilon} + \frac{\partial R}{\partial \omega}\right\}$$

$$\dot{\varepsilon} = \frac{1}{na^2}\left\{\frac{(1-e^2)^{\frac{1}{2}}}{e}\left[1-(1-e^2)^{\frac{1}{2}}\right]\frac{\partial R}{\partial e} - 2a\,\frac{\partial R}{\partial a} + \frac{1}{(1-e^2)^{\frac{1}{2}}\sin i}(1-\cos i)\ \frac{\partial R}{\partial i}\right\}$$

$$\dot{\Omega} = \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\ \partial R/\partial i \tag{6}$$

$$\dot{\omega} = \frac{1}{na^2}\left\{\frac{(1-e^2)^{\frac{1}{2}}}{e}\ \frac{\partial R}{\partial e} - \frac{\operatorname{ctg} i}{(1-e^2)^{\frac{1}{2}}}\ \partial R/\partial i\right\}$$

$$\dot{i} = \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\left\{\cos i\ \frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega} - (1-\cos i)\ \partial R/\partial \varepsilon\right\}$$

where $n = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}}$.

Equations 6 contain singularities at e = o and e = 1, (i.e. for both circular and parabolic orbits), and as such present computational difficulties for near circular or near parabolic trajectories. Singularities and the corresponding computational problems are also present when the plane of motion is in or near the equatorial plane. Consequently, Equations 6 are not suitable for trajectory calculations. The use of a new set of variables, formed by combining the variables used in Equations 6, will allow the expression of the set of equations in a form devoid of the circular singularity which is most important in low thrust trajectory considerations.

The new variables which allow the expression of the equations with no circular singularities are a, Ω, ε^*, σ, ξ, and ζ where

$$\sigma = e\cos\omega,\quad \zeta = (1-e^2)^{\frac{1}{2}}\cos i$$
$$\xi = e\sin\omega,\quad e^* = \varepsilon - \Omega \tag{7}$$

and a and Ω are the same as defined previously.

The $\partial R/\partial$ (old elements) in terms of $\partial R/\partial$ (new elements) are determined by utilizing the chain rule and evaluating the partial derivatives of the old elements with respect to the new from the relations of Equations 7.

This operations yields **

$$\left(\frac{\partial R}{\partial a}\right) = \partial R/\partial a$$

$$\left(\frac{\partial R}{\partial \varepsilon}\right) = \partial R/\partial \varepsilon^*$$

$$\left(\frac{\partial R}{\partial \Omega}\right) = \frac{\partial R}{\partial \Omega} - \frac{\partial R}{\partial \varepsilon^*} \tag{8}$$

$$\left(\frac{\partial R}{\partial e}\right) = \frac{\sigma}{e}\frac{\partial R}{\partial \sigma} + \frac{\xi}{e}\frac{\partial R}{\partial \xi} - \frac{e\zeta}{1-e^2}\frac{\partial R}{\partial \zeta}$$

$$\left(\frac{\partial R}{\partial \omega}\right) = \sigma\frac{\partial R}{\partial \xi} - \xi\frac{\partial R}{\partial \sigma}$$

$$\left(\frac{\partial R}{\partial i}\right) = -(1-e^2)^{\frac{1}{2}}\sin i\frac{\partial R}{\partial \zeta}$$

**

The parameter e is used throughout the remainder of this report to indicate $e = (\sigma^2 + \xi^2)^{\frac{1}{2}}$.

where the parentheses around a partial derivative, $(\frac{\partial R}{\partial a})$ etc., indicate the parameters in Equation 6. The time derivatives of the new elements in terms of the old are obtained from Equations 7 as:

$$\begin{aligned}
\dot{a} &= (\dot{a}) \\
\dot{\varepsilon}^* &= (\dot{\varepsilon}) - (\dot{\Omega}) \\
\dot{\Omega} &= (\dot{\Omega}) \\
\dot{\sigma} &= \frac{\sigma}{e}(\dot{e}) + (\dot{\omega})\,\xi \\
\dot{\xi} &= \frac{\xi}{e}(\dot{e}) + (\dot{\omega})\,\sigma \\
\dot{\zeta} &= \frac{-\,e\cos i}{(1-e^2)^{\frac{1}{2}}}(\dot{e}) - (\dot{i})(1-e^2)^{\frac{1}{2}}\sin i
\end{aligned} \tag{9}$$

Substitution of Equations 6, 7 and 8 into Equations 9 then yields the desired set of non-singular equations.

$$\begin{aligned}
\dot{a} &= \frac{2}{na}\,\partial R/\partial\varepsilon^* \\
\dot{\varepsilon}^* &= \frac{1}{na^2}\left\{\frac{(1-e^2)^{\frac{1}{2}}}{1+(1-e^2)^{\frac{1}{2}}}\left(\sigma\frac{\partial R}{\partial\sigma} + \xi\frac{\partial R}{\partial\xi}\right) + \zeta\frac{\partial R}{\partial\zeta} - 2a\frac{\partial R}{\partial a}\right\} \\
\dot{\Omega} &= -\frac{1}{na^2}\frac{\partial R}{\partial\zeta} \\
\dot{\sigma} &= \frac{1}{na^2}\left\{\frac{-(1-e^2)^{\frac{1}{2}}}{1+(1-e^2)^{\frac{1}{2}}}\,\sigma\frac{\partial R}{\partial\varepsilon^*} - (1-e^2)^{\frac{1}{2}}\frac{\partial R}{\partial\xi}\right\} \\
\dot{\xi} &= \frac{1}{na^2}\left\{\frac{-(1-e^2)^{\frac{1}{2}}}{1+(1-e^2)^{\frac{1}{2}}}\,\xi\frac{\partial R}{\partial\varepsilon^*} + (1-e^2)^{\frac{1}{2}}\frac{\partial R}{\partial\sigma}\right\} \\
\dot{\zeta} &= \frac{1}{na^2}\left\{\frac{\partial R}{\partial\Omega} - \zeta\frac{\partial R}{\partial\varepsilon^*}\right\}
\end{aligned} \tag{10}$$

The $R_{1,2}$ and $R_{2,1}$ in the three body equations, Equations 2, have precise definitions. However, in trajectory calculations only the partial derivatives $\partial R_r/\partial \bar{u}$ and $\partial R_\lambda/\partial \bar{q}$ are defined. The $\partial R/\partial$ (element) terms in Equations 10 are determined from these definitions by means of the chain rule. Hence, it is necessary to define $\bar{u}$ and $\bar{q}$ in terms of the elements $a, \varepsilon^*, \Omega, \sigma, \xi$ and ζ.

The orbital elements of a particular orbit may be defined in terms of a rectangular cartesian coordinate system in the plane of motion. The procedure used here is a slight deviation from the procedure given in References 5 and 6. A rectangular coordinate system may be defined with the x - axis toward the ascending node and the y - axis 90 degrees forward in the plane of motion. The radius vector $\bar{r}$ may then be expressed in terms of the orbital parameters a, σ, ζ and a generalized eccentric anomaly E which is defined (Ref. 6)

$$E - E_N = \left(\frac{GM}{a}\right)^{\frac{1}{2}} \int_{t_n}^{t} \frac{dt}{r} \tag{11}$$

where the subscript N denotes nodal passage. Restricting E such that its value is zero at nodal passage, the position vector $\bar{v}$ may be expressed

$$\bar{v} = (\bar{v}_N + \bar{e}a) \cos E + v_N \dot{\bar{v}}_N \left(\frac{a}{GM}\right)^{\frac{1}{2}} \sin E - \bar{e}a \tag{12}$$

where $\bar{e}$ is a vector of magnitude e directed toward the perihelion. Utilizing the relations (Reference 6)

$$\bar{v}_N = \begin{bmatrix} x_n \\ 0 \end{bmatrix} ; \quad \bar{e} = \begin{bmatrix} \sigma \\ \xi \end{bmatrix}$$

and

$$\dot{\bar{v}}_N = \begin{bmatrix} \dot{x}_n \\ \dot{y}_n \end{bmatrix} = \left(\frac{GM}{a}\right)^{\frac{1}{2}} \frac{1}{(1-e^2)^{\frac{1}{2}}} \begin{bmatrix} -\xi \\ 1 + \sigma \end{bmatrix}$$

The position vector may be expressed

$$\bar{v} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = a \begin{bmatrix} \left(1 - \frac{\xi^2}{1+\sigma}\right) \cos E - \frac{(1-e^2)^{\frac{1}{2}}\xi}{1+\sigma} \sin E - \sigma \\ \xi \cos E + (1-e^2)^{\frac{1}{2}} \sin E - \xi \\ 0 \end{bmatrix} \tag{13}$$

The generalized eccentric anomaly E is connected to the eccentric anomaly Ξ usually used, through the relation

$$E = \Xi + E^*$$

as shown in Figure 3. In the above procedure, E^* is chosen to make $E_N = 0$.

A vector in the x, y, axis system is related to a vector in the equatorial system through the direction cosine matrix A^*, where

$$A^* = \begin{bmatrix} S\Omega & C\Omega Ci & -C\Omega Si \\ C\Omega & -S\Omega Ci & S\Omega Si \\ 0 & -Si & -Ci \end{bmatrix} \tag{14}$$

the position vector in the equatorial plane is then expressed

$$\bar{\alpha} = \left[A^*\right] \bar{\beta} \tag{15}$$

where $\bar{\alpha}$ represents either $\bar{u}$ or $\bar{q}$ and $\bar{\beta}$ either $\bar{v}_r$ or $\bar{v}_\lambda$.
The $\partial R/\partial$ (element) terms then become

$$\begin{aligned}
\frac{\partial R}{\partial a} &= \frac{\partial R}{\partial \bar{\alpha}} \left\{ \left[A^*\right] \frac{\partial \bar{\beta}}{\partial a} \right\} \\
\frac{\partial R}{\partial \Omega} &= \frac{\partial R}{\partial \bar{\alpha}} \left\{ \left[\frac{\partial \left[A^*\right]}{\partial \Omega} \right] \bar{\beta} \right\} \\
\frac{\partial R}{\partial \varepsilon^*} &= \frac{\partial R}{\partial \bar{\alpha}} \left\{ \left[A^*\right] \frac{\partial \bar{\beta}}{\partial \varepsilon^*} \right\} \\
\frac{\partial R}{\partial \sigma} &= \frac{\partial R}{\partial \bar{\alpha}} \left\{ \left[A^*\right] \frac{\partial \bar{\beta}}{\partial \sigma} + \left[\frac{\partial \left[A^*\right]}{\partial i} \frac{\partial i}{\partial \sigma} \right] \bar{\beta} \right\} \\
\frac{\partial R}{\partial \xi} &= \frac{\partial R}{\partial \bar{\alpha}} \left\{ \left[A^*\right] \frac{\partial \bar{\beta}}{\partial \xi} + \left[\frac{\partial \left[A^*\right]}{\partial i} \frac{\partial i}{\partial \xi} \right] \bar{\beta} \right\} \\
\frac{\partial R}{\partial \zeta} &= \frac{\partial R}{\partial \bar{\alpha}} \left\{ \left[\frac{\partial \left[A^*\right]}{\partial i} \frac{\partial i}{\partial \zeta} \right] \bar{\beta} \right\}
\end{aligned} \tag{16}$$

The partials of $[A^*]$ required in Equations 16 are

$$\frac{\partial [A^*]}{\partial i} = \begin{bmatrix} S\Omega & -C\Omega Si & -C\Omega Ci \\ C\Omega & S\Omega Si & S\Omega Ci \\ 0 & -Ci & Si \end{bmatrix} \tag{17a}$$

and

$$\frac{\partial [A^*]}{\partial \Omega} = \begin{bmatrix} C\Omega & -S\Omega Ci & S\Omega Si \\ -S\Omega & -C\Omega Ci & C\Omega Si \\ 0 & 0 & 0 \end{bmatrix} \tag{17b}$$

The partials of $\bar{\beta}$ required in Equations 16 are

$$\frac{\partial \bar{\beta}}{\partial a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{18a}$$

where

$$\alpha_1 = (1-\frac{\xi^2}{1+\sigma})CE - \frac{(1-e^2)^{\frac{1}{2}}}{1+\sigma}\xi\, SE - \sigma - \left\{(\frac{\xi^2}{1+\sigma} - 1)SE - \frac{(1-e^2)^{\frac{1}{2}}}{1+\sigma}\xi\, CE\right\}$$

$$X\left\{\frac{3[E(1+\sigma) - (\sigma+e^2)SE - \xi(1-e^2)(1-CE)]}{2[1+\sigma-(\sigma+e^2)CE - \xi(1-e^2)^{\frac{1}{2}} SE]}\right\}$$

$$\alpha_2 = \xi\, CE + (1-e^2)^{\frac{1}{2}} SE - \xi - \left\{(1-e^2)^{\frac{1}{2}} CE - \xi\, SE\right\}$$

$$X\left\{\frac{3E[(1+\sigma) - (\sigma+e^2)SE - \xi(1-e^2)^{\frac{1}{2}}(1-CE)]}{2[1+\sigma-(\sigma+e^2)CE - \xi(1-e^2)^{\frac{1}{2}} SE]}\right\}$$

$$\alpha_3 = 0$$

$$\frac{\partial \bar{\beta}}{\partial \epsilon^*} = \frac{a}{1+\sigma-(\sigma+e^2)CE - \xi(1-e^2)^{\frac{1}{2}}SE}\begin{bmatrix} (\xi^2-\sigma-1)\, SE - \xi(1-e^2)^{\frac{1}{2}} CE \\ (1+\sigma)\left[(1-e^2)^{\frac{1}{2}} CE - \xi\, SE\right] \\ 0 \end{bmatrix} \tag{18b}$$

$$\frac{\partial \bar{\beta}}{\partial \sigma} = a \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \tag{18c}$$

where

$$\gamma_1 = \frac{\xi^2 CE + \xi(1-e^2)^{\frac{1}{2}} SE - (1+\sigma)^2}{(1+\sigma)^2} + \frac{\sigma\, \xi\, SE}{(1+\sigma)(1-e^2)^{\frac{1}{2}}} + \frac{(\xi^2 - \sigma - 1) SE - \xi\, (1-e^2)^{\frac{1}{2}} CE}{1+\sigma}$$

$$X \; \frac{\left[(1+\sigma)^2 - \xi^2\right](1-e^2)^{\frac{1}{2}} SE - \xi\, (1+\sigma-\xi^2)(1-CE)}{(1+\sigma)(1-e^2)^{\frac{1}{2}}\left[1+\sigma - (\sigma+e^2)CE - (1-e^2)^{\frac{1}{2}} SE\right]}$$

$$\gamma_2 = -\frac{\sigma\, SE}{(1-e^2)^{\frac{1}{2}}} + \left\{(1-e^2)^{\frac{1}{2}} CE - \xi\, SE\right\}$$

$$X \left\{ \frac{\left[(1+\sigma)^2 - \xi^2\right](1-e^2)^{\frac{1}{2}} SE - \xi\,(1+\sigma-\xi^2)(1-CE)}{(1+\sigma)\ (1-e^2)^{\frac{1}{2}}\left[1+\sigma - (\sigma+e^2)CE - (1-e^2)^{\frac{1}{2}} SE\right]} \right\}$$

$$\gamma_3 = 0$$

$$\frac{\partial \bar{\beta}}{\partial \xi} = a \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \tag{18d}$$

where

$$\delta_1 = \frac{\xi^2\, SE}{(1+\sigma)(1-e^2)^{\frac{1}{2}}} - \frac{(1-e^2)^{\frac{1}{2}} SE + 2\xi CE}{1+\sigma} + \left\{ \frac{(\xi^2 - \sigma - 1) SE - \xi (1-e^2)^{\frac{1}{2}} CE}{1+\sigma} \right\}$$

$$X \left\{ \frac{(1-e^2-\xi^2)\ (1-CE) + 2\xi\ (1-e^2)^{\frac{1}{2}}\ SE}{(1-e^2)^{\frac{1}{2}}\left[1+\sigma - (\sigma+e^2)CE - \xi(1-e^2)^{\frac{1}{2}} SE\right]} \right\}$$

$$\delta_2 = (CE-1) - \frac{\xi\, SE}{(1-e^2)^{\frac{1}{2}}} + \left\{(1-e^2)^{\frac{1}{2}} CE - \xi SE\right\}$$

$$X \left\{ \frac{(1-e^2-\xi^2)\ (1-CE) + 2\xi\ (1-e^2)^{\frac{1}{2}}\ SE}{(1-e^2)^{\frac{1}{2}}\left[1+\sigma - (\sigma+e^2)CE - \xi\,(1-e^2)^{\frac{1}{2}} SE\right]} \right\}$$

$$\delta_3 = 0$$

Finally, the partials of i required in Equations 16 are

$$\frac{\partial i}{\partial \sigma} = - \frac{\sigma}{1-e^2} \text{ ctg } i$$

$$\frac{\partial i}{\partial \xi} = - \frac{\xi}{1-e^2} \text{ ctg } i \tag{19}$$

$$\frac{\partial i}{\partial \zeta} = - \frac{1}{(1-e^2)^{\frac{1}{2}} \sin i}$$

The two sets of $\partial R/\partial$ (element) terms in Equations 10 are then

$$\frac{\partial R_r}{\partial a_r} = \frac{F}{mp} \bar{q} \left[A^*\right]_r \left[\frac{\partial}{\partial a_r} \bar{v}_r\right]$$

$$\frac{\partial R_r}{\partial \Omega_r} = \frac{F}{mp} \bar{q} \left[\frac{\partial}{\partial \Omega_r} \left[A^*\right]_r\right] \bar{v}_r$$

$$\frac{\partial R_r}{\partial \varepsilon^*_r} = \frac{F}{mp} \bar{q} \left[A^*\right]_r \left[\frac{\partial}{\partial \varepsilon^*_r} \bar{v}_r\right] \tag{20a}$$

$$\frac{\partial R_r}{\partial \sigma_r} = \frac{F}{mp} \bar{q} \left\{\left[A^*\right]_r \left[\frac{\partial}{\partial \sigma_r} \bar{v}_r\right] + \left[\frac{\partial}{\partial i_r} \left[A^*\right]_r \frac{\partial i_r}{\partial \sigma_r}\right] \bar{v}_r\right\}$$

$$\frac{\partial R_r}{\partial \xi_r} = \frac{F}{mp} \bar{q} \left\{\left[A^*\right]_r \left[\frac{\partial}{\partial \xi_r} \bar{v}_r\right] + \left[\frac{\partial}{\partial i_r} \left[A^*\right]_r \frac{\partial i_r}{\partial \xi_r}\right] \bar{v}_r\right\}$$

$$\frac{\partial R}{\partial \zeta_r} = \frac{F}{mp} \bar{q} \left[\frac{\partial}{\partial i_r} \left[A^*\right]_r \frac{\partial i_r}{\partial \zeta_r}\right] \bar{v}_r$$

and

$$\frac{\partial R_\lambda}{\partial a_\lambda} = - GM \left[\left(\frac{1}{r^3} - \frac{1}{p^3}\right) \bar{q} - \frac{3}{r^5} (\bar{u} \cdot \bar{q}) \bar{u} \right] \left[A^*\right]_\lambda \left[\frac{\partial}{\partial a_\lambda} \bar{v}_\lambda\right]$$

$$\frac{\partial R_\lambda}{\partial \Omega_\lambda} = - GM \left[\left(\frac{1}{r^3} - \frac{1}{p^3}\right) \bar{q} - \frac{3}{r^5} (\bar{u} \cdot \bar{q}) \bar{u} \right] \left[\frac{\partial}{\partial \Omega_\lambda} \left[A^*\right]_\lambda\right] \bar{v}_\lambda$$

$$\frac{\partial R_\lambda}{\partial \epsilon^*_\lambda} = - GM \left[\left(\frac{1}{r^3} - \frac{1}{p^3}\right) \bar{q} - \frac{3}{r^5} (\bar{u} \cdot \bar{q}) \bar{u} \right] \left[A^*\right]_\lambda \left[\frac{\partial}{\partial \epsilon^*_\lambda} \bar{v}_\lambda\right]$$

$$\frac{\partial R_\lambda}{\partial \sigma_\lambda} = - GM \left[\left(\frac{1}{r^3} - \frac{1}{p^3}\right) \bar{q} - \frac{3}{r^5} (\bar{u} \cdot \bar{q}) \bar{u} \right] \left\{ \left[A^*\right]_\lambda \left[\frac{\partial}{\partial \sigma_\lambda} \bar{v}_\lambda\right] + \left[\frac{\partial}{\partial i_\lambda} \left[A^*\right]_\lambda \frac{\partial i_\lambda}{\partial \sigma_\lambda}\right] \bar{v}_\lambda \right\} \quad (20b)$$

$$\frac{\partial R_\lambda}{\partial \xi_\lambda} = - GM \left[\left(\frac{1}{r^3} - \frac{1}{p^3}\right) \bar{q} - \frac{3}{r^5} (\bar{u} \cdot \bar{q}) \bar{u} \right] \left\{ \left[A^*\right]_\lambda \left[\frac{\partial}{\partial \xi_\lambda} \bar{v}_\lambda\right] + \left[\frac{\partial}{\partial i_\lambda} \left[A^*\right]_\lambda \frac{\partial i_\lambda}{\partial \xi_\lambda}\right] \bar{v}_\lambda \right\}$$

$$\frac{\partial R_\lambda}{\partial \zeta_\lambda} = - GM \left[\left(\frac{1}{r^3} - \frac{1}{p^3}\right) \bar{q} - \frac{3}{r^5} (\bar{u} \cdot \bar{q}) \bar{u} \right] \left[\frac{\partial}{\partial i_\lambda} \left[A^*\right]_\lambda \frac{\partial i_\lambda}{\partial \zeta_\lambda}\right] \bar{v}_\lambda$$

The partial derivatives on the right hand side of Equations 20a and 20b are evaluated from Equations 17, 18 and 19.

The complete form of the equations of motion could be formulated by the substitution of Equations 20a and 20b, evaluated through Equations 17, 18 and 19, into the respective sets of Equations 10.

Space considerations preclude presentation of the complete form. It may be noted however, that the only singularities in Equations 10 are contained in the $\partial R/\partial$ (element) terms. An examination of the right hand sides of Equations 20, in conjunction with Equations 13, 17, 18 and 19, indicates that $\partial R_r/\partial \sigma_r$, $\partial R_r/\partial \xi_r$ and $\partial R_r/\partial \zeta_r$ contain singularities at $i_r = 0$ and $e_r = 1$ and that $\partial R_\lambda/\partial \sigma_\lambda$, $\partial R_\lambda/\partial \xi_\lambda$ and $\partial R_\lambda/\partial \zeta_\lambda$ contain singularities at $i_\lambda = 0$ and $e_\lambda = 1$. However, the circular singularities, $e_r = 0$ and $e_\lambda = 0$, present in Equations 6 have been removed through the transformation. This form of the equations should then have some usefulness in the calculation of low thrust trajectories that traverse orbits which are circular or elliptical and do not lie in the equatorial plane.

PRELOAD RELATIONS

Initial conditions for a trajectory calculation will usually be given in the plumbline domain. It will therefore be necessary to perform the following preload calculations to specify the initial conditions in terms of the orbital elements.

1. Position and velocity vectors in equatorial axis system:

$$\bar{u}_o = \left[\phi_o\right]_1 \left[A_o - 90^o\right]_2 \bar{x}_{po}$$

$$\dot{\bar{u}}_o = \left[\phi_o\right]_1 \left[A_o - 90^o\right]_2 \dot{\bar{x}}_{po}$$

Lagrange multipliers in equatorial axis system:

$$\bar{q}_o = \left[\phi_o\right]_1 \left[A_o - 90^o\right]_2 \bar{\lambda}_o$$

$$\dot{\bar{q}}_o = \left[\phi_o\right]_1 \left[A_o - 90^o\right]_2 \dot{\bar{\lambda}}_o$$

2. Specific angular momentum, $\bar{H}$

$$\bar{H}_{r_o} = \bar{u}_o \times \dot{\bar{u}}_o$$

$$\bar{H}_{\lambda o} = \bar{q}_o \times \dot{\bar{q}}_o$$

3. Specific energy constant, k

$$k_{ro} = \frac{1}{2} \dot{u}_o^{\,2} - GM/u_o$$

$$k_{\lambda o} = \frac{1}{2} \dot{q}_o^{\,2} - GM/q_o$$

4. Semi-major axis, a

$$a_{ro} = -\ GM/2k_{ro}$$

$$a_{\lambda o} = -\ GM/2k_{\lambda o}$$

5. Semi-latus rectum (orbital parameter), ℓ

$$\ell_{ro} = H_{ro}^{2}/GM$$

$$\ell_{\lambda o} = H_{\lambda o}^{2}/GM$$

6. Eccentricity, e

$$e_{ro} = (1 - \ell_{ro}/a_{ro})^{\frac{1}{2}}$$

$$e_{\lambda o} = (1 - \ell_{\lambda o}/a_{\lambda o})^{\frac{1}{2}}$$

7. Eccentric anomaly, Ξ

$$\Xi_{ro} = \cos^{-1}\left(\frac{a_{ro} - u_o}{a_{ro}\ e_{ro}}\right)$$

$$\Xi_{\lambda o} = \cos^{-1}\left(\frac{a_{\lambda o} - q_o}{a_{\lambda o}\ e_{\lambda o}}\right)$$

8. True anomaly from perigee, ν

$$\nu_{ro} = 2\tan^{-1}\left(\frac{1 + e_{ro}}{1 - e_{ro}}\right)^{\frac{1}{2}} \tan \frac{1}{2}\ \Xi_{ro}$$

$$\nu_{\lambda o} = 2\tan^{-1}\left(\frac{1 + e_{\lambda o}}{1 - e_{\lambda o}}\right)^{\frac{1}{2}} \tan \frac{1}{2}\ \Xi_{\lambda o}$$

9. Postion angle from line of nodes, θ

A vector along the line of nodes in the direction of the ascending node, $\bar{N}$, is determined by crossing a vector normal to the equatorial reference plane into a vector normal to the plane of motion, $\bar{H}$.

$$\bar{N}_o = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} X \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{bmatrix} H_2 \\ -H_1 \\ 0 \end{bmatrix}_o$$

A vector perpendicular to the line nodes, $\bar{M}$, is determined by crossing the angular momentum vector, $\bar{H}$, into the line of nodes vector $\bar{N}$.

$$\bar{M}_o = \bar{H}_o \text{ X } \bar{N}_o = \begin{bmatrix} H_1 H_3 \\ H_2 H_3 \\ -(H_1^2 + H_2^2) \end{bmatrix}_o$$

Taking the scalar product $\bar{N}_o$ with the position vector gives

$$\bar{N}_{ro} \cdot \bar{u}_o = N_{ro}\, u_o \cos \theta_{ro} = u_{1o}\, H_{2ro} - u_{2o}\, H_{1ro}$$

$$\bar{N}_{\lambda o} \cdot \bar{q}_o = N_{\lambda o}\, q_o \cos \theta_{\lambda o} = q_{1o}\, H_{2\lambda o} - q_{2o}\, H_{1\lambda o}$$

Taking the scalar product of $\bar{M}_o$ with the position vector gives

$$\bar{M}_{ro} \cdot \bar{u}_o = M_{ro}\, u_o \sin \theta_{ro} = (u_{1o}\, H_{1ro} + u_{2o}\, H_{2ro})H_{3ro} - u_{3o}(H_{1ro}^2 + H_{2ro}^2)$$

$$\bar{M}_{\lambda o} \cdot \bar{q}_o = M_{\lambda o}\, q_o \sin \theta_{\lambda o} = (q_{1o}\, H_{1\lambda o} + q_{2o}\, H_{2\lambda o})H_{3\lambda o} - q_{3o}(H_{1\lambda o}^2 + H_{2\lambda o}^2)$$

from whence

$$\theta_{ro} = \tan^{-1} \frac{(u_{1o}\, H_{1ro} + u_{2o}\, H_{2ro})H_{3ro} - u_{3o}(H_{1ro}^2 + H_{2ro}^2)}{(u_{1o}\, H_{2ro} - u_{2o}\, H_{1ro})H_{ro}}$$

$$\theta_{\lambda o} = \tan^{-1} \frac{(q_{1o}\, H_{1\lambda o} + q_{2o}\, H_{2\lambda o})H_{3\lambda o} - q_{3o}(H_{1\lambda o}^2 + H_{2\lambda o}^2)}{(q_{1o}\, H_{2\lambda o} - q_{2o}\, H_{1\lambda o})H_{\lambda o}}$$

10. Argument of perigee, ω

$$\omega_{ro} = \theta_{ro} - \nu_{ro}$$

$$\omega_{\lambda o} = \theta_{\lambda o} - \nu_{\lambda o}$$

11. Parameter σ

$$\sigma_{ro} = e_{ro} \cos \omega_{ro}$$

$$\sigma_{\lambda o} = e_{\lambda o} \cos \omega_{\lambda o}$$

12. Parameter ξ

$$\xi_{ro} = e_{ro} \sin \omega_{ro}$$

$$\xi_{\lambda o} = e_{\lambda o} \sin \omega_{\lambda o}$$

13. Inclination angle, i

Taking the scalar product of the angular momentum vector $\bar{H}$ with a vector normal to the equatorial plane

$$\bar{H} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = H \cos i = -H_3$$

The scalar product of the vector normal to the line of nodes with a vector normal to the equatorial plane gives

$$\bar{M} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = M \cos (90-i) = H_1^2+H_2^2$$

From whence

$$i_{ro} = \tan^{-1} - \frac{\left(H_{1ro}^2 + H_{2ro}^2\right)^{\frac{1}{2}}}{H_{3ro}}$$

$$i_{\lambda o} = \tan^{-1} - \frac{\left(H_{1\lambda o}^2 + H_{2\lambda o}^2\right)^{\frac{1}{2}}}{H_{3\lambda o}}$$

14. Parameter, ζ

$$\zeta_{ro} = (1-e_{ro}^2)^{\frac{1}{2}} \cos i_{ro}$$

$$\zeta_{\lambda o} = (1-e_{\lambda o}^2)^{\frac{1}{2}} \cos i_{\lambda o}$$

15. Generalized eccentric anomaly, E

$$E_{ro} = 2 \tan^{-1} \frac{(1 + e^2{}_{ro})^{\frac{1}{2}} \tan \frac{1}{2}\theta_{ro}}{1 + \sigma_{ro} + \xi_{ro} \tan \frac{1}{2}\theta_{ro}}$$

$$E_{\lambda o} = 2 \tan^{-1} \frac{(1 - e^2{}_{\lambda o})^{\frac{1}{2}} \tan \frac{1}{2}\theta_{\lambda o}}{1 + \sigma_{ro} + \xi_{ro} \tan \frac{1}{2}\theta_{ro}}$$

16. Mean argument from line of nodes at epoch, ε^*

$$\varepsilon^*_{ro} = 0$$

$$\varepsilon^*_{\lambda o} = 0$$

17. Epoch time of nodal passage, t_{No}

$$t_{Nro} = t_o - E_{ro} + \frac{\sigma_{ro} + e_{ro}^2}{1 + \sigma_{ro}} \sin E_{ro} + \frac{(1-e_{ro}^2)^{\frac{1}{2}}}{1 + \sigma_{ro}} \xi_{ro} \, (1-\cos E_{ro})$$

$$t_{N\lambda o} = t_o - E_{\lambda o} + \frac{\sigma_{\lambda o} + e_{\lambda o}^2}{1 + \sigma_{\lambda o}} \sin E_{\lambda o} + \frac{(1-e_{\lambda o}^2)^{\frac{1}{2}}}{1 + \sigma_{\lambda o}} \xi_{\lambda o} \, (1-\cos E_{\lambda o})$$

18. Longitude of ascending node, Ω

The scalar product of a vector along the u_2 axis with the line of nodes yields

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \bar{N} = N \cos \Omega = - H_1$$

The scalar product of a vector along the u_1 axis with the line of nodes yields

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \bar{N} = N \cos (90-\Omega) = H_2$$

Hence

$$\Omega_{ro} = - \frac{H_{2\,ro}}{H_{1\,ro}}$$

$$\Omega_{\lambda o} = - \frac{H_{2\lambda o}}{H_{1\lambda o}}$$

Relations 4, 11, 12, 14, 16 and 18 give the required initial conditions for calculations in the orbital elements.

COMPUTATIONAL METHOD

Assuming the Runge-Kutta integration procedure will be used, the procedure presented below may be used to perform claculations going from the jth time step to the (j + 1) time step. The more cumbersome relations presented in the previous sections of this report will be referred to in this section with the reference followed by an asterisk (*).

From step j, the following quantities are known:

a_r, Ω_r, ε^*_r, σ_r, ξ_r, ζ_r, u_1, u_2, u_3, x_{p_1}, x_{p_2}, x_{p_3}, ω_r,

a_λ, Ω_λ, ε^*_λ σ_λ, ξ_λ, ζ_λ, q_1, q_2, q_3, λ_1, λ_2, λ_3, ω_λ, i_λ,

i_r, e_r, $\left[A^*\right]_r$, E_r, x_r, y_r, n_r

e_λ, $\left[A^*\right]_\lambda$, E_λ, x_λ, y_λ, n_λ

1. Compute: $\dfrac{\partial\left[A^*\right]_r}{\partial i_r}$ and $\dfrac{\partial\left[A^*\right]_\lambda}{\partial i_\lambda}$ (Equation 17a)*

2. Compute: $\dfrac{\partial\left[A^*\right]_r}{\partial \Omega_r}$ and $\dfrac{\partial\left[A^*\right]_\lambda}{\partial \Omega_\lambda}$ (Equation 17b)*

3. Compute. $\dfrac{\partial \overline{v}_r}{\partial a_r}$ and $\dfrac{\partial \overline{v}_\lambda}{\partial a_\lambda}$ (Equation 18a)*

4. Compute: $\dfrac{\partial \overline{v}_r}{\partial \varepsilon^*_r}$ and $\dfrac{\partial \overline{v}_\lambda}{\partial \varepsilon^*_\lambda}$ (Equation 18b)*

5. Compute: $\dfrac{\partial \overline{v}_r}{\partial \sigma_r}$ and $\dfrac{\partial \overline{v}_\lambda}{\partial \sigma_\lambda}$ (Equation 18c)*

6. Compute: $\dfrac{\partial \overline{v}_r}{\partial \xi_r}$ and $\dfrac{\partial \overline{v}_\lambda}{\partial \xi_\lambda}$ (Equation 18d)*

7. Compute: $\dfrac{\partial i_r}{\partial \sigma_r} = - \dfrac{\sigma_r}{1-e_r^2} \text{ ctg } i_r$

$$\frac{\partial i_\lambda}{\partial \sigma_\lambda} = - \frac{\sigma_\lambda}{1-e_\lambda^2} \text{ ctg } i_\lambda$$

8. Compute: $$\frac{\partial i_r}{\partial \xi_r} = - \frac{\xi_r}{1-e_r^2} \operatorname{ctg} i_r$$

$$\frac{\partial i_\lambda}{\partial \xi_\lambda} = - \frac{\xi_\lambda}{1-e_\lambda^2} \operatorname{ctg} i_\lambda$$

9. Compute: $$\frac{\partial i_r}{\partial \zeta_r} = - \frac{1}{(1-e_r^2)^{\frac{1}{2}} \sin i_r}$$

$$\frac{\partial i_\lambda}{\partial \zeta_\lambda} = - \frac{1}{(1-e_\lambda^2)^{\frac{1}{2}} \sin i_\lambda}$$

10. Compute:

$$\bar{v}_r = \begin{bmatrix} x_r \\ y_r \\ 0 \end{bmatrix}$$

$$\bar{v}_\lambda \quad \begin{bmatrix} x_\lambda \\ y_\lambda \\ 0 \end{bmatrix}$$

11. Compute: $$m_j = m_{j-1} - \dot{m}\,(t_j - t_{j-1})$$

12. Compute: $$\frac{\partial R_r}{\partial a_r}, \frac{\partial R_r}{\partial \Omega_r}, \frac{\partial R_r}{\partial \varepsilon_r^*}, \frac{\partial R_r}{\partial \sigma_r}, \frac{\partial R_r}{\partial \xi_r}, \frac{\partial R_r}{\partial \zeta_r}$$ (Equation 20a)*

13. Compute: $$\frac{\partial R_\lambda}{\partial a_\lambda}, \frac{\partial R_\lambda}{\partial \Omega_\lambda}, \frac{\partial R_\lambda}{\partial \varepsilon_\lambda^*}, \frac{\partial R_\lambda}{\partial \sigma_\lambda}, \frac{\partial R_\lambda}{\partial \xi_\lambda}, \frac{\partial R_\lambda}{\partial \zeta_\lambda}$$ (Equation 20b)*

14. Compute:

$$\dot{a}_r = \frac{2}{n_r a_r} \frac{\partial R_r}{\partial \varepsilon_r^*}$$

$$\dot{\varepsilon}_r^* = \frac{1}{n_r a_r^2} \left\{ \frac{(1-e_r^2)^{\frac{1}{2}}}{1+(1-e_r^2)^{\frac{1}{2}}} \left[\sigma_r \frac{\partial R_r}{\partial \sigma_r} + \xi_r \frac{\partial R_r}{\partial \xi_r} \right] + \zeta_r \frac{\partial R_r}{\partial \zeta_r} - 2a_r \frac{\partial R_r}{\partial a_r} \right\}$$

$$\dot{\Omega}_r = - \frac{1}{n_r a_r^2} \frac{\partial R_r}{\partial \zeta_r}$$

$$\dot{\sigma}_r = \frac{-1}{n_r a_r^2} \left\{ \frac{(1-e_r^2)^{\frac{1}{2}} \sigma_r}{1+(1-e_r^2)^{\frac{1}{2}}} \frac{\partial R_r}{\partial \varepsilon_r^*} + (1-e_r^2)^{\frac{1}{2}} \frac{\partial R_r}{\partial \xi_r} \right\}$$

$$\dot{\xi}_r = \frac{-1}{n_r a_r^2} \left\{ \frac{(1-e_r^2)^{\frac{1}{2}} \xi_r}{1+(1-e_r 2)^{\frac{1}{2}}} \frac{\partial R_r}{\partial \varepsilon_r^*} - (1-e_r^2)^{\frac{1}{2}} \frac{\partial R_r}{\partial \sigma_r} \right\}$$

$$\dot{\zeta}_r = \frac{1}{n_r a_r^2} \left\{ \frac{\partial R_r}{\partial \Omega_r} - \zeta_r \frac{\partial R_r}{\partial \varepsilon_r^*} \right\}$$

15. Integrate each relation of step 16 to obtain the values of a_r, ε_r^* , Ω_r, σ_r, ξ_r, and ζ_r at time j + 1.

16. Compute:

$$\dot{a}_\lambda = \frac{2}{n_\lambda a_\lambda} \frac{\partial R_\lambda}{\partial \varepsilon_\lambda^*}$$

$$\dot{\varepsilon}_\lambda^* = \frac{1}{n_\lambda a_\lambda^2} \left\{ \frac{(1-e_\lambda^2)^{\frac{1}{2}}}{1+(1-e_\lambda^2)^{\frac{1}{2}}} \left[\sigma_\lambda \frac{\partial R_\lambda}{\partial \sigma_\lambda} + \xi_\lambda \frac{\partial R_\lambda}{\partial \xi_\lambda} \right] + \zeta_\lambda \frac{\partial R_\lambda}{\partial \zeta_\lambda} - 2a_\lambda \frac{\partial R_\lambda}{\partial a_\lambda} \right\}$$

$$\dot{\Omega}_\lambda = \frac{-1}{n_\lambda a_\lambda^2} \frac{\partial R_\lambda}{\partial \zeta_\lambda}$$

$$\dot{\sigma}_\lambda = \frac{-1}{n_\lambda a_\lambda^2} \left\{ \frac{(1-e_\lambda^2)^{\frac{1}{2}} \sigma_\lambda}{1+(1-e_\lambda^2)^{\frac{1}{2}}} \frac{\partial R_\lambda}{\partial \varepsilon_\lambda^*} + (1-e_\lambda^2)^{\frac{1}{2}} \frac{\partial R_\lambda}{\partial \xi_\lambda} \right\}$$

$$\dot{\xi}_\lambda = \frac{-1}{n_\lambda a_\lambda^2} \left\{ \frac{(1-e_\lambda^2)^{\frac{1}{2}} \xi_\lambda}{1+(1-e_\lambda^2)^{\frac{1}{2}}} \frac{\partial R_\lambda}{\partial \varepsilon_\lambda^*} - (1-e_\lambda^2)^{\frac{1}{2}} \frac{\partial R_\lambda}{\partial \sigma_\lambda} \right\}$$

$$\dot{\zeta}_\lambda = \frac{1}{n_\lambda a_\lambda^2} \left\{ \frac{\partial R_\lambda}{\partial \Omega_\lambda} - \zeta_\lambda \frac{\partial R_\lambda}{\partial \varepsilon_\lambda^*} \right\}$$

17. Integrate each relation of step 16 to obtain values of a_λ, ε_λ^* Ω_λ , σ_λ, ξ_λ and ζ_λ at time j + 1.

18. Compute for time j + 1:

$$n_r = (GM/a_r{}^3)^{\frac{1}{2}}$$

$$n_\lambda = (GM/a_\lambda{}^3)^{\frac{1}{2}}$$

19. Iterate for E_r at time j + 1, from:

$$n_r(t-t_{Nor}) + \varepsilon_r^* = E_r - \frac{\sigma_r+e_r{}^2}{1+\sigma_r} \sin E_r - \frac{(1-e_r{}^2)^{\frac{1}{2}}}{1+\sigma_r} \xi_r (1-\cos E_r)$$

20. Iterate for E_λ at time j+ 1, from:

$$n_\lambda (t-t_{Nc\lambda}) + \varepsilon_\lambda^* = E_\lambda - \frac{\sigma_\lambda+e_\lambda{}^2}{1+\sigma_\lambda} \sin E_\lambda - \frac{(1-e_\lambda{}^2)^{\frac{1}{2}}}{1+\sigma_\lambda} \xi_\lambda (1-\cos E_\lambda)$$

21. Compute for time j + 1:

$$e_r = (\sigma_r{}^2 + \xi_r{}^2)^{\frac{1}{2}}$$

$$e_\lambda = (\sigma_\lambda{}^2 + \xi_\lambda{}^2)^{\frac{1}{2}}$$

22. Compute for time j + 1:

$$\omega_r = \tan^{-1} (\xi_r/\sigma_r)$$

$$\omega_\lambda = \tan^{-1} (\xi_\lambda/\sigma_\lambda)$$

23. Compute for time j + 1:

$$i_r = \cos^{-1} \frac{\zeta_r}{(1-e_r{}^2)^{\frac{1}{2}}}$$

$$i_\lambda = \cos^{-1} \frac{\zeta_\lambda}{(1-e_\lambda{}^2)^{\frac{1}{2}}}$$

24. Compute for time j + 1:

$$x_r = a_r \left[\left(1 - \frac{\xi_r^2}{1+\sigma_r}\right) \cos E_r - \frac{(1-e_r^2)^{\frac{1}{2}} \xi_r}{1+\sigma_r} \sin E_r - \sigma_r \right]$$

$$x_\lambda = a_\lambda \left[\left(1 - \frac{\xi_\lambda^2}{1+\sigma_\lambda}\right) \cos E_\lambda - \frac{(1-e_\lambda^2)^{\frac{1}{2}} \xi_\lambda}{1+\sigma_\lambda} \sin E_\lambda - \sigma_\lambda \right]$$

25. Compute for time j + 1:

$$y_r = a_r \left[\xi_r \cos E_r + (1-e_r^2)^{\frac{1}{2}} \sin E_r - \xi_r \right]$$

$$y_\lambda = a_\lambda \left[\xi_\lambda \cos E_\lambda + (1-e_\lambda^2)^{\frac{1}{2}} \sin E_\lambda - \xi_\lambda \right]$$

26. Compute for time j + 1:

$$\left[A^*\right]_r = \begin{bmatrix} S\Omega_r & C\Omega_r Ci_r & -C\Omega_r Si_r \\ C\Omega_r & -S\Omega_r Ci_r & S\Omega_r Si_r \\ 0 & -Si_r & -Ci_r \end{bmatrix}$$

$$\left[A^*\right]_\lambda = \begin{bmatrix} S\Omega_\lambda & C\Omega_\lambda Ci_\lambda & -C\Omega_\lambda Si_\lambda \\ C\Omega_\lambda & -S\Omega_\lambda Ci_\lambda & S\Omega_\lambda Si_\lambda \\ 0 & -Si_\lambda & -Ci_\lambda \end{bmatrix}$$

27. Compute for time j + 1:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \left[A^*\right]_r \begin{bmatrix} x_r \\ y_r \\ 0 \end{bmatrix}$$

28. Compute for time j + 1:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \left[A^*\right]_\lambda \begin{bmatrix} x_\lambda \\ y_\lambda \\ 0 \end{bmatrix}$$

29. Compute for time j + 1:

$$\begin{bmatrix} x_{1p} \\ x_{2p} \\ x_{3p} \end{bmatrix} = \left[\left[\phi_o\right]_1 \left[A_o - 90^\circ\right]\right]^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

30. Compute for time j + 1:

$$\begin{bmatrix} \lambda 1 \\ \lambda 2 \\ \lambda 3 \end{bmatrix} = \left[\left[\phi_o \right]_1 \left[A_o - 90^o \right]_2 \right]^T \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

31. Compute for time j + 1:

$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ 0 \end{bmatrix} = - \frac{n_r a_r^2}{(x_r^2 + y_r^2)^{\frac{1}{2}}} \left\{ \begin{bmatrix} \frac{1-e_r^2}{1+\sigma_r} + \sigma_r \\ \xi_r \\ 0 \end{bmatrix} \sin E_r + \frac{(1-e_r^2)^{\frac{1}{2}}}{1+\sigma_r} \begin{bmatrix} \xi_r \\ -(1+\sigma_r) \\ 0 \end{bmatrix} \right\}$$

32. Compute for time j + 1:

$$\begin{bmatrix} \dot{x}_\lambda \\ \dot{y}_\lambda \\ 0 \end{bmatrix} = - \frac{n_\lambda a_\lambda^2}{(x_\lambda^2 + y_\lambda^2)^{\frac{1}{2}}} \left\{ \begin{bmatrix} \frac{1-e_\lambda^2}{1+\sigma_\lambda} + \sigma_\lambda \\ \xi_\lambda \\ 0 \end{bmatrix} \sin E_\lambda + \frac{(1-e_\lambda^2)^{\frac{1}{2}}}{1+\sigma_\lambda} \begin{bmatrix} \xi_\lambda \\ -(1+\sigma_\lambda) \\ 0 \end{bmatrix} \right\}$$

33. Compute for time j + 1:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \left[A^* \right]_r \begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ 0 \end{bmatrix}$$

34. Compute for time j + 1:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \left[A^* \right]_\lambda \begin{bmatrix} \dot{x}_\lambda \\ \dot{y}_\lambda \\ 0 \end{bmatrix}$$

35. Compute for time j + 1:

$$\begin{bmatrix} \dot{x}_{p1} \\ \dot{x}_{p2} \\ \dot{x}_{p3} \end{bmatrix} = \left[\left[\phi_o \right]_1 \left[A_o - 90^o \right]_2 \right]^T \begin{bmatrix} \dot{u}_1 \\ u_2 \\ \dot{u}_3 \end{bmatrix}$$

36. Compute for time j + 1:

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} = \begin{bmatrix} [\phi_\alpha]_1 \, [A_o - 90^\circ]_2 \end{bmatrix}^T \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

It may be seen from the above procedures, that all quantities required for the next time step, plus the parameters x_{p_1}, x_{p_2}, x_{p_3}, $\dot{x}_{p_1}$, $\dot{x}_{p_2}$, $\dot{x}_{p_3}$, λ_1, λ_2, λ_3, $\dot{\lambda}_1$, $\dot{\lambda}_2$, and $\dot{\lambda}_3$ are calculated.

REFERENCES

(1) Cox, J. Grady and Shaw, W. A., "Preliminary Investigations on Three Dimensional Optimum Trajectories", Progress Report No. 1 on Studies in the Fields of Space Flight and Guidance Theory. NASA Report MTP-AERO-61-91. Dec. 18, 1961.

(2) Miner, W. E. "Methods for Trajectory Computation", Aeroballistics Internal Note No. 3-61. May 10, 1961.

(3) Moulton, F. R. "An Introduction to Celestial Mechanics", Second Edition, The Macmillan Company, New York, 1914.

(4) Brouwer, Dirk and Clemence, G.M. "Methods of Celestial Mechanics", Academic Press, New York, 1961.

(5) Sperling, H. J. "On the Computation of Keplerian Ellipses", ABMA Report No. DA-TM-48-60, May 27, 1960.

(6) Sperling, H. J. "On the Computation of Keplerian Conic Sections", NASA Report No. MTP-AERO-61-19, May 10, 1961.

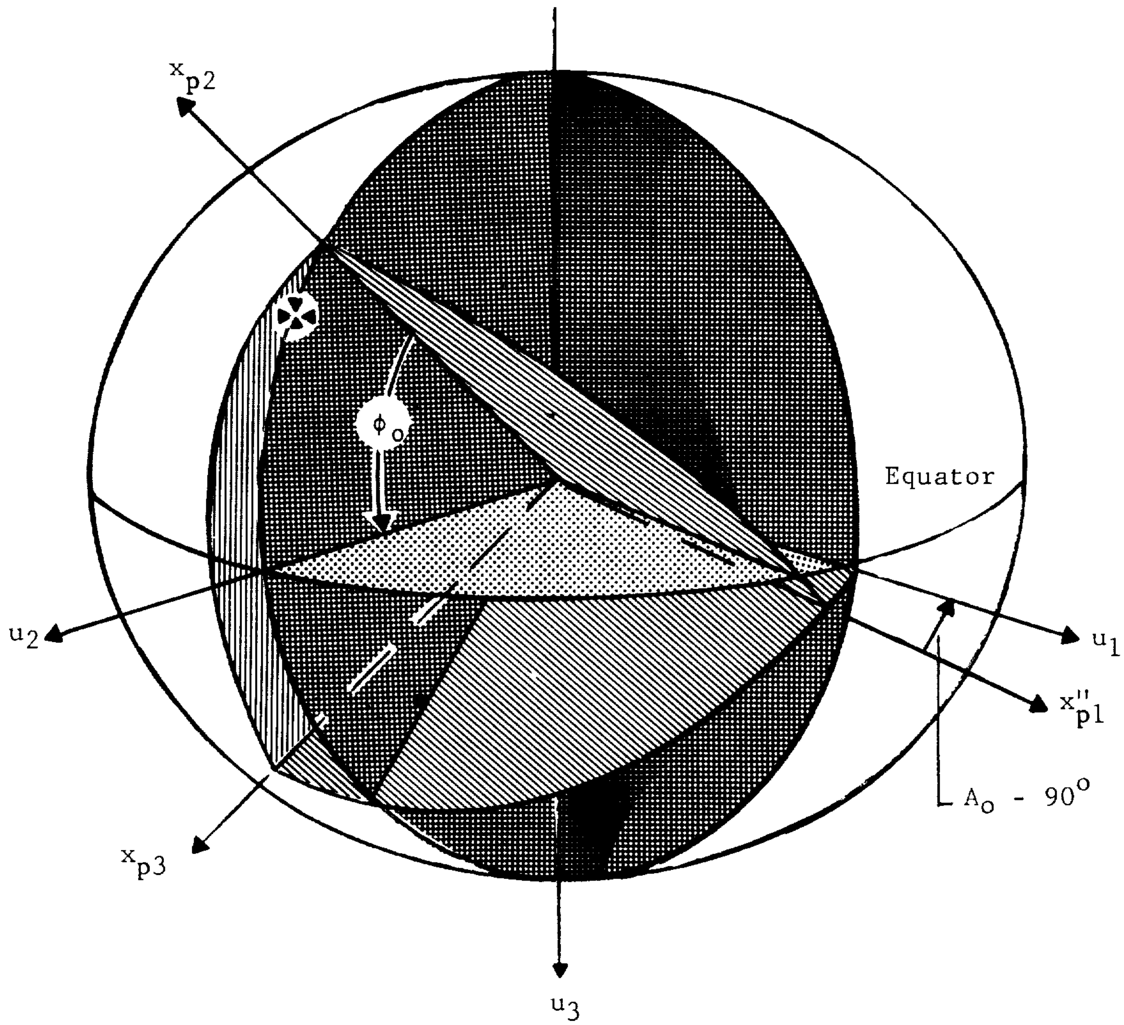

FIGURE 1. EQUATORIAL SYSTEM

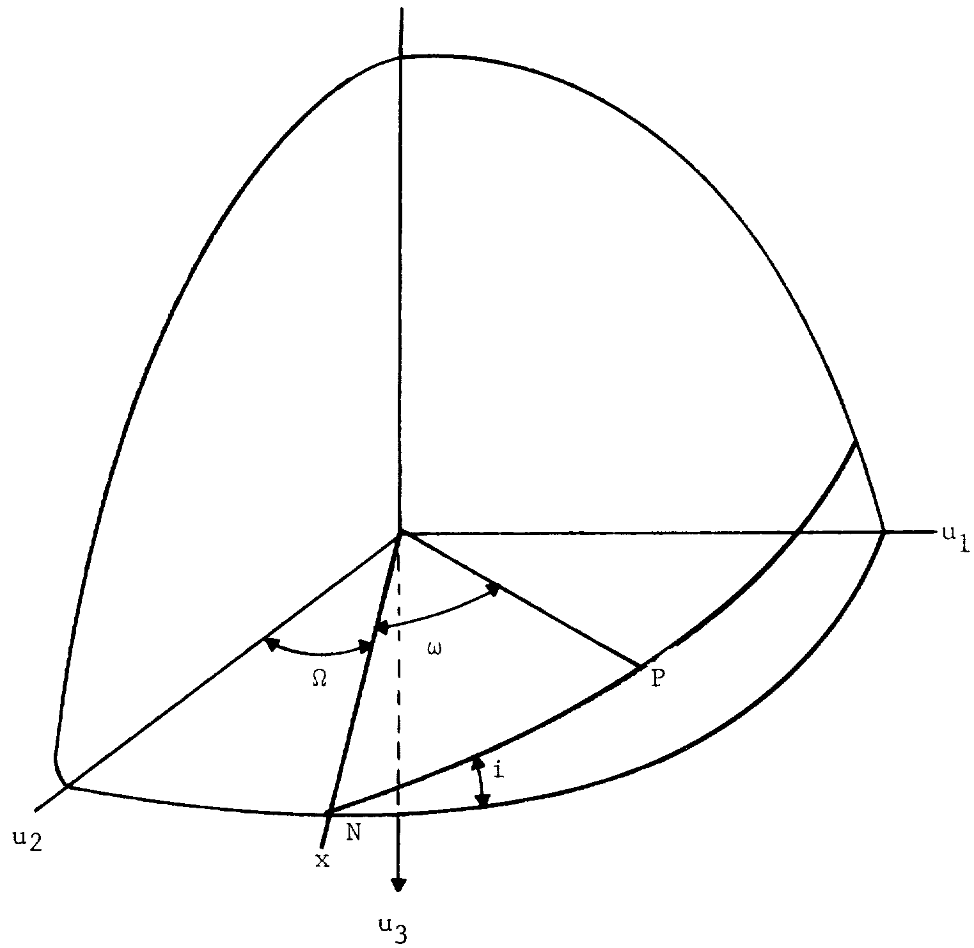

FIGURE 2. ORBITAL - EQUATORIAL ANGLES

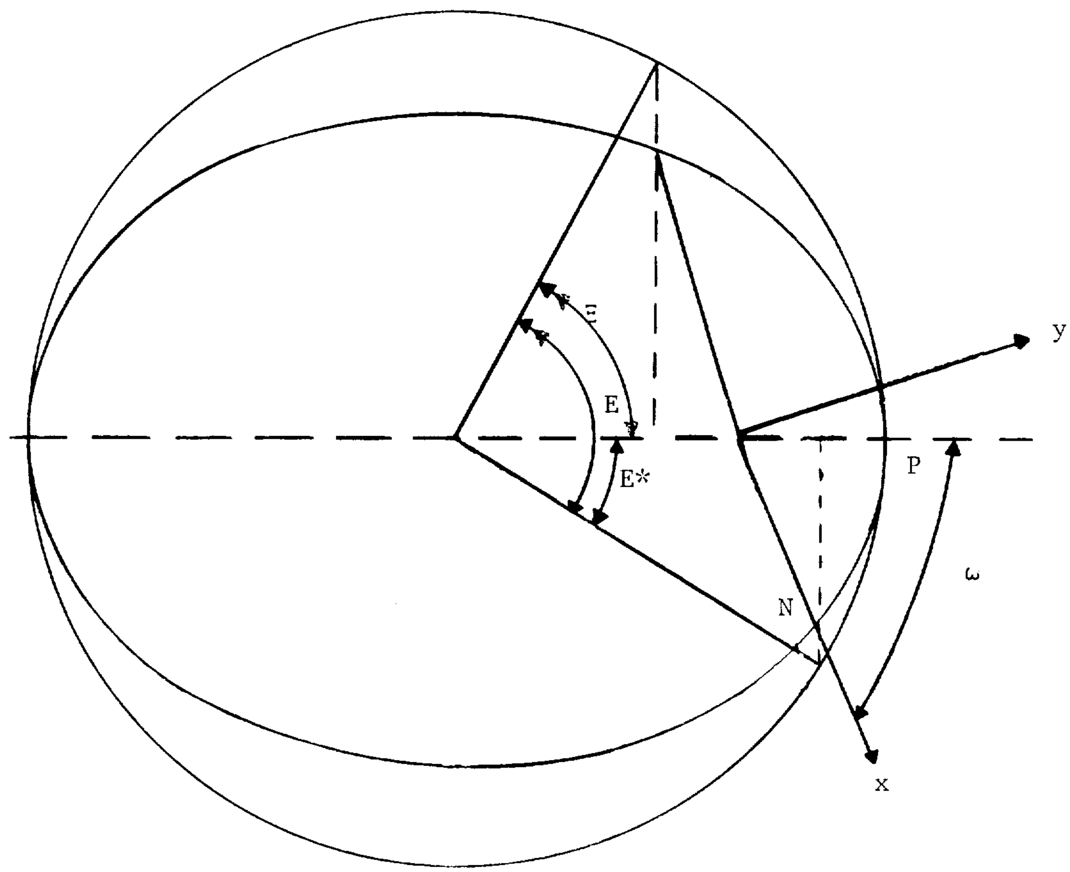

FIGURE 3. ECCENTRIC ANOMALY RELATIONS

ROCKET BOOSTER VERTICAL CLIMB

OPTIMALITY

by

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ABSTRACT

This paper discusses the problem of the optimum burning program for the vertical climb of a rocket. This problem is of engineering interest in view of its applicability to the case of sounding rockets or to the study of the vertical climb of a rocket prior to pitch-over maneuver. The essential objectives of this work, from the standpoint of both physical and variational aspects, are:

I. A study of the optimum burning program based on a generalized model. That is, assuming an arbitrary aerodynamic configuration and an arbitrary atmospheric scheme.

II. An analysis of the numerical solution of the boundary-value variational problem. In particular, the determination of corner points and the integration of an admissible set of adjoint variables for the case where the aerodynamic drag is of the form $D = D(v, h)$.

In this paper a generalized expression for the optimum burning program (or control variable program), valid for any arbitrary aerodynamic characteristics and any arbitrary atmospheric model assumed, is derived. A numerical method for determining the position of the corner points, for cases where $D = D(v, h)$, is discussed. Numerical examples are included.

The case of maximum final altitude is analyzed in the applications presented showing the numerical integration of the variable multipliers along the extremal, as well as the switching function $H_{\tilde{\beta}}(\tau)$. Thus, a practical example of the numerical treatment of the Euler equations and

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corner point determination, which is of value as a basic model for the understanding of more sophisticated problems not affording closed-form solutions, is given.

LIST OF SYMBOLS

C_{D_o}	Zero-lift drag coefficient
D	Aerodynamic drag (lb.)
g	Acceleration of gravity (ft. sec^{-2})
G	Functional to be minimized
h	Flight altitude (ft.)
H	Hamiltonian
m	Mass of the rocket (lb. sec^{2} ft^{-1})
m_o	Initial mass (lb. sec^{2} ft^{-1})
q	Generalized coordinate
S	Reference surface (ft.^{2})
t	Time (sec.)
V	Flight velocity (ft. sec^{-1})
V_e	Velocity of the gases at the exit section of the nozzle (ft. sec^{-1})
z	Dimensionless velocity
β	Mass flow (lb. sec. ft^{-1})
η	Density ratio
λ_o	Constant Lagrange multiplier
μ	Variable Lagrange multiplier
ν	Canomical variable
ρ	Atmospheric density (lb. sec^{2} ft^{-4})
τ	Dimensionless time

List of Symbols - Continued

ψ	End-constraints
Ψ	Equation of terminal variation
Ω	Euler-Lagrange sum

Superscripts

$$(\cdots)' = \frac{d}{d\tau}(\cdots)$$

$$\widetilde{(\cdots)} = \text{Dimensionless quantity}$$

Subscripts

$$(\cdots)_I = \text{Initial condition}$$

$$(\cdots)_F = \text{Final condition}$$

1. EQUATIONS OF MOTION AND VARIATIONAL PROBLEM

The equations of motion, referred to a cartesian system fixed to a flat Earth, are written

$$f_1 \equiv z' + \frac{\tilde{D}(z,\tilde{h}) - \tilde{\beta}}{\tilde{m}} + 1 = 0 \tag{1}$$

$$f_2 \equiv \tilde{h}' - z = 0 \tag{2}$$

$$f_3 \equiv \tilde{m}' + \tilde{\beta} = 0 \tag{3}$$

Where the following dimensionless variables have been included;

$$\tilde{\beta} = \frac{\beta V_e}{m_o g} \quad , \quad \tau = \frac{t g}{V_e} \quad , \quad \tilde{h} = \frac{h g}{V_e^2}$$

$$z = \frac{V}{V_e} \quad , \quad \tilde{D} = \frac{D}{m_o g} \quad , \quad \tilde{m} = \frac{m}{m_o}$$

The aerodynamic drag of the rocket will be expressed in the following general form:

$$\tilde{D} = k \eta z^2 C_{D_o}(z,\tilde{h}) = \tilde{D}(z,\tilde{h}) \tag{4}$$

where

$$\eta = \rho/\rho_o = \eta(\tilde{h}) \quad , \quad k = \frac{1}{2} \frac{\rho_o V_e^2 S}{m_o g}$$

It is assumed that the velocity of the gases at the exit section of the nozzle, V_e, and the acceleration of gravity, g, are constants. Any solution of the set of equations (1) to (3), is expressed in terms of the state variables, $z(\tau)$, $\tilde{h}(\tau)$, $\tilde{m}(\tau)$ and the control variable, $\tilde{\beta}(\tau)$. The problem under discussion here is that of "finding the optimum solution $z(\tau)$, $\tilde{h}(\tau)$, $\tilde{m}(\tau)$, $\tilde{\beta}(\tau)$ of Eqs. (1) to (3), satisfying given boundary conditions of the form,

$$\psi_\rho\left(z_I, \tilde{h}_I, \tilde{m}_I, \tau_I, z_F, \tilde{h}_F, \tilde{m}_F, \tau_F\right) = 0 \quad , \rho = 1, \ldots, r \leqq 7 \tag{5}$$

and minimizing a generalized functional of the terminal values

$$G = G\left(z_I, \tilde{h}_I, \tilde{m}_I, \tau_I, z_F, \tilde{h}_F, \tilde{m}_F, \tau_F\right) \text{''} . \tag{6}$$

In the development which follows the generalized state variables will be denoted q_i, $i = 1; \ldots; 3$. Thus,

$$q_1 = z \quad , \quad q_2 = \tilde{h} \quad , \quad q_3 = \tilde{m}$$

The Euler-Lagrange sum is

$$\Omega = \mu_i f_i \quad , \quad i = 1, \ldots, 3 \tag{7}$$

The canonical variables $(\tau, q_i, \tilde{\beta}, \nu_i)$ related to $(\tau, q_i, q_i', \tilde{\beta}, \mu_i)$ by the equations

$$\nu_i = \Omega_{q_i'}(\tau, q_i, q_i', \tilde{\beta}, \mu_i) \quad , \quad 0 = f_i(\tau, q_i, q_i', \tilde{\beta}) \tag{8}$$

are now introduced. Applying the Legendre transformation of the variational problem into canonical form

$$\nu_i = \Omega_{q_i'} \qquad , \qquad \nu_i q_i' - \Omega = H \tag{9}$$

and forming the Fundamental Function

$$F(\tau, q_i, \tilde{\beta}, \nu_i) = \nu_i \frac{dq_i}{d\tau} - H \tag{10}$$

the following canonical equations of the extremals may be derived (Refs. 1, 5 and 10)

$$\nu_i' + H_{q_i} = 0 \tag{11}$$

$$q_i' - H_{\nu_i} = 0 \tag{12}$$

Eqs. (12) are the equations of motion while Eqs. (11), in this case, are identical with the Euler equations since $\nu_i = \Omega_{q_i'} = \mu_i$, as derived from Eqs. (1) to (3) and (7). Assuming that the control variable is bounded, i.e., $\tilde{\beta}_{min.} \leqq \tilde{\beta} \leqq \tilde{\beta}_{max.}$, the following equations associated with different admissible control variations $\delta\tilde{\beta}$ (restricted or one-sided admissible control variations and unrestricted or both-sided admissible control variations) may also be obtained

$$\left.\begin{aligned}
&a)\ \frac{\partial H}{\partial \tilde{\beta}} = 0 \quad \text{for} \quad \delta\tilde{\beta} \gtreqless 0\ ,\ \tilde{\beta}_{min} \leqq \tilde{\beta} \leqq \tilde{\beta}_{max.}\ ,\ \tilde{\beta}\text{-variable} \\
&b)\ \frac{\partial H}{\partial \tilde{\beta}} \leqq 0 \quad \text{for} \quad \delta\tilde{\beta} \geqq 0\ , \quad \tilde{\beta} = \tilde{\beta}_{min.} = const. \\
&c)\ \frac{\partial H}{\partial \tilde{\beta}} \geqq 0 \quad \text{for} \quad \delta\tilde{\beta} \leqq 0\ , \quad \tilde{\beta} = \tilde{\beta}_{max.} = const.
\end{aligned}\right\} \tag{13}$$

From Eqs. (1), (2), (3), (7), (9) and (11) it follows that

$$\mu_1' - \frac{\mu_1}{\tilde{m}}\tilde{D}_z + \mu_2 = 0 \tag{14}$$

$$\mu_2' - \frac{\mu_1}{\tilde{m}}\tilde{D}_{\tilde{h}} = 0 \tag{15}$$

$$\mu_3' + \frac{\mu_1}{\tilde{m}^2}(\tilde{D} - \tilde{\beta}) = 0 \tag{16}$$

Also, from Eqs. (1) to (3) and (9)

$$H = \left(\frac{\mu_1}{\tilde{m}} - \mu_3\right)\tilde{\beta} - \mu_1\left(\frac{\tilde{D}}{\tilde{m}} + 1\right) + \mu_2 z \tag{17}$$

and therefore

$$H_{\tilde{\beta}} = \frac{\mu_1}{\tilde{m}} - \mu_3 \tag{18}$$

Eqs. (1) to (3), (13) [(a), (b) or (c) according to the admissible control variations] and (14) to (16) constitute a set of first necessary conditions for an extremal. Every admissible solution, $z(\tau)$, $\tilde{h}(\tau)$, $\tilde{m}(\tau)$, $\tilde{\beta}(\tau)$, $\mu_1(\tau)$, $\mu_2(\tau)$, $\mu_3(\tau)$ of the preceding set,belongs to an extremal. This set is determined since we have 7 equations in 7 variables. An admissible extremal must satisfy other necessary conditions in addition to being a solution of the previous set. We will now consider these necessary conditions.

From the parametric formulation of the variational problem (Refs. 1 and 5) we have

$$\frac{d}{d\tau}\left[\Omega - q_i' \, \Omega_{q_i'}\right] = \frac{\partial \Omega}{\partial \tau} \tag{19}$$

Since in our case $\Omega_\tau = 0$, it follows that

$$\left(\mu_3 - \frac{\mu_1}{\tilde{m}}\right)\tilde{\beta} + \mu_1\left(\frac{\tilde{D}}{\tilde{m}} + 1\right) - \mu_2 z = C = const. \tag{20}$$

along the extremal. Eq. (20) is a consequence of the Euler equations (14) to (16) and may replace any one of them if desired.

1.1 Weierstrass Condition and Maximality Principle

The Weierstrass condition requires that at any point on the extremal

$$W = \Delta\Omega - \Delta q_i' \, \Omega_{q_i'} \geqq 0 \tag{21}$$

and since

$$\Delta\Omega = \nu_i \, \Delta q_i' - \Delta H \tag{22}$$

then Eq. (21) leads to

$$W = \nu_i \, \Delta q_i' - \Delta H - \Delta q_i' \, \Omega_{q_i'} = - \Delta H \geqq 0$$

Consequently,

$$\Delta H = H(\tau, q_i, \tilde{\beta}^* + \Delta\tilde{\beta}, \nu_i) - H(\tau, q_i, \tilde{\beta}^*, \nu_i) \leqq 0 \qquad (23)$$

where $\tilde{\beta}^*$ is the optimum control. Eq. (23) is the canonical form of the Weierstrass condition, and implies that for any admissible set (τ, q_i, ν_i) the optimum control $\tilde{\beta}^*$ is that which maximizes the Hamiltonian H. This condition, applicable for bounded or unbounded control $\tilde{\beta}$, is known as the Maximality Principle.

From Eqs. (13) [(a), (b) and (c)], (17) and (23) it is seen that in the present case the Hamiltonian is linear in the control variable and that the conditions previously discussed may be graphically represented as indicated in Fig. 1.

1.2 Non-Singularity

An extremal arc of class D' (Ref. 3) will be called non-singular if along each sub-arc the determinant

$$\mathcal{D} = \begin{vmatrix} \Omega_{q_k' \mu_i} & \Omega_{q_k' q_i'} \\ 0 & f_{k \, q_i'} \end{vmatrix}, \quad k, i = 1, .., 3, \qquad (24)$$

is different from zero. Then, along each sub-arc, q_i' and μ_i will be continuously differentiable. For our problem the determinant (24) is

$$\mathcal{D} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \qquad (25)$$

Thus, any admissible extremal solution obtained will be non-singular.

1.3 Transversality Condition

In addition to the necessary conditions for an extremal just discussed, we can determine from first variation arguments, that at terminal points of the extremal, the following sub-conditions of Transversality must be satisfied

$$\left(\lambda_o\, G_{z_\alpha} \mp \mu_{1_\alpha}\right) dz_\alpha = 0 \quad , \quad \left(\lambda_o\, G_{\tilde{m}_\alpha} \mp \mu_{3_\alpha}\right) d\tilde{m}_\alpha = 0$$

$$\left(\lambda_o\, G_{\tilde{h}_\alpha} \mp \mu_{2\alpha}\right) d\tilde{h}_\alpha = 0 \quad , \quad \left(\lambda_o\, G_{\tau_\alpha} \mp C\right) d\tau_\alpha = 0 \tag{26}$$

In Eqs. (26) the (−) sign is applied when $\alpha = I$ and the (+) sign is applied when $\alpha = F$. Eq. (26) must be satisfied for any admissible set $\left(dq_{i_I}\, ,\, dq_{i_F}\, ,\, d\tau_I\, ,\, d\tau_F\right) \neq (0, 0, 0, 0)$ consistent with the equations of terminal variations derived from Eq. (5) as

$$\Psi_\rho \equiv \psi_{q_I}\, dq_I + \psi_{q_F}\, dq_F + \psi_{\tau_I}\, d\tau_I + \psi_{\tau_F}\, d\tau_F = 0, \; \rho = 1, .., r \leq 7, \tag{27}$$

For a normal extremal, as assumed in this case, there exists a unique set of variable multipliers. The constant Lagrange multiplier λ_o, in Eq. (26), may then be taken $\lambda_o = 1$.

1.4 Erdmann-Weierstrass Corner Conditions

Eqs. (11) and (19) hold at any point on the extremal arc $E(I,F)$. Now, Eq. (11) indicates that at a corner, even if H_{q_i} is discontinuous, the canonical variables are continuous, i.e., $(\nu_i)^- = (\nu_i)^+$. The (−) and (+) signs indicate limiting values approaching the corner from the left and from the right side. Similar reasoning can be applied to Eq. (19). Thus, the following continuity conditions may be derived:

$$(\mu_i)^- = (\mu_i)^+ \quad , \quad C^- = C^+ \tag{28}$$

Consequently, from the preceding continuity conditions it follows that at the corners

$$\left(H_{\tilde{\beta}} \right)^{-} = \left(H_{\tilde{\beta}} \right)^{+} \tag{29}$$

if we assume that all of the state variables are continuous along the extremal (e.g., it is assumed that at corners only the mass flow may be discontinuous and that there is no staging). Conditions (13) and (29) lead us to the conclusion that at any corner point [i.e., point at which $\tilde{\beta}(\tau)$ is discontinuous], $\partial H / \partial \tilde{\beta} = 0$. Based on this conclusion, the function $H_{\tilde{\beta}}[\tau, q_i(\tau), \tilde{\beta}(\tau), \nu_i(\tau)] = H_{\tilde{\beta}}(\tau)$ will be called the switching function since for

a) $H_{\tilde{\beta}}(\tau) \geqq 0 \quad , \quad \tilde{\beta} = \tilde{\beta}_{max.}$

b) $H_{\tilde{\beta}}(\tau) = 0 \quad , \quad \tilde{\beta}$ – var. or CORNER

c) $H_{\tilde{\beta}}(\tau) \leqq 0 \quad , \quad \tilde{\beta} = \tilde{\beta}_{min.}$

thus indicating when the mode of control shifts from one to another form along the extremal.

2. GENERALIZED EXPRESSION FOR THE OPTIMUM BURNING PROGRAM ALONG THE $\tilde{\beta}$ - VAR. SUB-ARC.

As seen from Eqs. (13a), (18) and (20), along the $\tilde{\beta}$ - var. sub-arc, the following equations must hold

$$H_{\tilde{\beta}} = \frac{\mu_1}{\tilde{m}} - \mu_3 = 0 \tag{30}$$

$$\mu_1\left(\frac{\tilde{D}}{\tilde{m}} + 1\right) - \mu_2 z = C \tag{31}$$

Thus, from total differentiation of Eq. (31) and accounting for Eqs. (1) to (3) and (14) to (16), it may be obtained

$$\frac{\mu_1}{\tilde{m}}\left(\frac{\partial \tilde{D}}{\partial z} + \tilde{D}\right) - \mu_2 = 0 \tag{32}$$

Eqs. (31) and (32) lead to

$$\mu_1\left[z\left(\tilde{D}_z + \tilde{D}\right) - \tilde{D} - \tilde{m}\right] + C\tilde{m} = 0 \tag{33}$$

After total differentiation of Eq. (33) and using the preceding relations, we obtain

$$\left[z\left(\tilde{D}_z + \tilde{D}\right) - \tilde{D} - \tilde{m}\right]\left(\frac{\mu_1}{\tilde{m}}\tilde{D}_z - \mu_2\right) + \mu_1\left[z\left(\tilde{D}_{zz} + \tilde{D}_z\right) + \tilde{D}\right]\left(\frac{\tilde{\beta} - \tilde{D}}{\tilde{m}} - 1\right)$$

$$+ \mu_1\left[z\left(\tilde{D}_{z\tilde{h}} + \tilde{D}_{\tilde{h}}\right) - \tilde{D}_{\tilde{h}}\right]z - \left(C - \mu_1\right)\tilde{\beta} = 0 \tag{34}$$

Introducing the function

$$\varphi = z\left(\tilde{D}_z + \tilde{D}\right) - \tilde{D} - \tilde{m} \tag{35}$$

it follows from Eq. (33)

$$\mu_1 = -\frac{C\,\tilde{m}}{\varphi} \tag{36}$$

Replacing Eq. (36) in Eq. (34), rearranging and eliminating C, we see that the generalized optimum control program or generalized burning program is explicitly given by

$$\tilde{\beta} = \frac{\tilde{D}\varphi + \left[\tilde{D} + z\left(\tilde{D}_{zz} + \tilde{D}_z\right)\right]\left(\tilde{D} + \tilde{m}\right) - \tilde{m}z\left[\left(\tilde{D}_{z\tilde{h}} + \tilde{D}_{\tilde{h}}\right)z - \tilde{D}_{\tilde{h}}\right]}{z\left(\tilde{D} + 2\tilde{D}_z + \tilde{D}_{zz}\right)} \tag{37}$$

The control program obtained in Eq. (37) is valid for any atmospheric scheme adopted as well as any arbitrary aerodynamic characteristics assumed. As will be shown, the control program $\tilde{\beta} = \tilde{\beta}(z, \tilde{h}, \tilde{m})$ given in Eq. (37) is applicable for given-time or for free-time problems (i.e., for $C \neq 0$ or $C = 0$). In fact, for problems where $C = 0$, the compatibility condition of Eqs. (31) and (32) is

$$\begin{vmatrix} \dfrac{\tilde{D}}{\tilde{m}} + 1 & -z \\ \dfrac{\tilde{D}_z + \tilde{D}}{\tilde{m}} & -1 \end{vmatrix} = 0 \tag{38}$$

which implies $\varphi = 0$. Consequently, for free-time problems $(C = 0)$, the optimum burning program along the $\tilde{\beta}$ - var. sub-arc may be readily obtained from the generalized expression in Eq. (37) as

$$\tilde{\beta} = \frac{\left[\tilde{D} + z\left(\tilde{D}_{zz} + \tilde{D}_z\right)\right]\left(\tilde{D} + \tilde{m}\right) - \tilde{m}\, z\left[z\left(\tilde{D}_{z\tilde{h}} + \tilde{D}_{\tilde{h}}\right) - \tilde{D}_{\tilde{h}}\right]}{z\left(\tilde{D} + 2\,\tilde{D}_z + \tilde{D}_{zz}\right)} \tag{39}$$

Eq. (39) may be easily verified by taking the total differential

$$\frac{d}{d\tau}\ \varphi\left(z, \tilde{h}, \tilde{m}\right) = 0 \tag{40}$$

3. TYPICAL VARIATIONAL PROBLEMS AND BOUNDARY CONDITIONS

During the vertical ascent of a sounding rocket or during the phase of vertical ascent of a rocket booster prior to the initiation of the pitch-over maneuver, different optimal requrements may be imposed. For example, the rocket may be required to climb up to a certain final altitude with minimum propellant expenditure as it carries scientific equipment of maximum possible weight, or for a given propellant weight a maximum final altitude may be desired. In other cases, it may be desireable to attain maximum energy per unit mass when the given amount of propellant is expended (end of the vertical climb phase of a rocket booster) before starting the pitch-over maneuver. This may be of interest when large paylaods are put into orbit. At any rate, due to the zero-length launching conditions (vertical launching from rest) of large rockets, for ballistic or orbital missions, a necessary first phase of vertical climb, to which the problem treated in this paper is applicable, will always be required. During this phase it may be of interest to optimize a certain functional which is specified according to the optimal mission criteria chosen. Since this is a matter of the particular case considered and of the optimality criteria decided upon, we will only consider in the following work some examples of optimal problems that may be derived from the generalized formulation

$$G = G\left(z_I, \tilde{h}_I, \tilde{m}_I, \tau_I, z_F, \tilde{h}_F, \tilde{m}_F, \tau_F\right) = min.$$

and to which the results previously obtained apply. The object here is to show how additional natural end-conditions may be derived depending on the minimal problem proposed, and boundary conditions imposed.

3.1 Maximum Altitude, Free-Time, Given Propellant Mass

For this case the following boundary conditions will be assumed given

$$
\begin{aligned}
&\psi_1 \equiv z_I - a_I = 0 \qquad &&\psi_4 \equiv \tau_I - c_I = 0 \\
&\psi_2 \equiv \tilde{h}_I - b_I = 0 \qquad &&\psi_5 \equiv z_F - d_F = 0 \\
&\psi_3 \equiv \tilde{m}_I - 1 = 0 \qquad &&\psi_6 \equiv \tilde{m}_F - e_F = 0
\end{aligned}
\tag{41}
$$

$$a_I\ ,\ b_I\ ,\ c_I\ ,\ d_F\ ,\ e_F = const.$$

The function to be minimized is now

$$G = \tilde{h}_I - \tilde{h}_F = b_I - \tilde{h}_F = min. \tag{42}$$

Consequently, from the Transversality Condition and the boundary conditions (41) it may be found that

$$
\begin{aligned}
(\mu_{2_F} - 1)\, d\tilde{h}_F = 0 \quad &, \quad d\tilde{h}_F \neq 0 \quad \therefore\ \mu_{2_F} = 1 \\
C\, d\tau_F = 0 \quad &, \quad d\tau_F \neq 0 \quad \therefore\ C = 0 = const.
\end{aligned}
\tag{43}
$$

3.2 Minimum Fuel Consumption, Free-Time, Given Final Altitude

Assume, in this case, that the following boundary conditions are given

$$\Psi_1 \equiv z_I - a_I = 0 \qquad \Psi_4 \equiv \tau_I - c_I = 0$$

$$\Psi_2 \equiv \tilde{h}_I - b_I = 0 \qquad \Psi_5 \equiv \tilde{h}_F - d_F = 0 \tag{44}$$

$$\Psi_3 \equiv \tilde{m}_I - 1 = 0 \qquad a_I, b_I, c_I, d_F = const.$$

The function to be minimized is

$$G = \tilde{m}_I - \tilde{m}_F = 1 - \tilde{m}_F = min. \tag{45}$$

Thus, from the Transversality Condition, and conditions (44) and (45), the following natural boundary conditions follow

$$\mu_{1_F} \, dz_F = 0 \,, \quad dz_F \neq 0 \quad \therefore \quad \mu_{1_F} = 0$$

$$(\mu_{3_F} - 1) \, d\tilde{m}_F = 0 \,, \quad d\tilde{m}_F \neq 0 \quad \therefore \quad \mu_{3_F} = 1 \tag{46}$$

$$C \, d\tau_F = 0 \,, \quad d\tau_F \neq 0 \quad \therefore \quad C = 0 = const.$$

Consequently, at the final point F it may be obtained that

$$\mu_{2_F} = \left(\frac{\tilde{\beta}}{z}\right)_F \tag{47}$$

Eq. (47) shows that for $z_F \neq 0$, μ_2 vanish at the final point if the arrival at this point is performed through a coasting sub-arc. For given time problems $C \neq 0$ and then

$$\mu_{2_F} = \left(\frac{\tilde{\beta} - C}{z}\right)_F \tag{48}$$

3.3 Maximum Final Energy Per Unit Mass, Free-Time, Given Fuel Consumption

The following boundary conditions are now assumed

$$\begin{aligned}
\psi_1 &\equiv z_I - a_I = 0 & \qquad \psi_4 &\equiv \tau_I - c_I = 0 \\
\psi_2 &\equiv \tilde{h}_I - b_I = 0 & \qquad \psi_5 &\equiv \tilde{m}_F - d_F = 0 \\
\psi_3 &\equiv \tilde{m}_I - 1 = 0 & \qquad & a_I, b_I, c_I, d_F = const.
\end{aligned} \tag{49}$$

The function to be minimized is

$$G = \left[\tilde{m}\left(\frac{1}{2}z^2 + \tilde{h}\right)\right]_F^I = const. - d_F\left(\frac{z_F^2}{2} + \tilde{h}_F\right) = min.$$

which assuming a normal extremal is equivalent to the optimal problem

$$G = -\left(\frac{1}{2} z_F^2 + \tilde{h}_F\right) = min. \tag{50}$$

Thus, the following natural boundary conditions may be obtained

$$\begin{aligned}
&\left(\mu_{1_F} - z_F\right) dz_F = 0 \ , \quad dz_F \neq 0 \quad \therefore \ \mu_{1_F} = z_F \\
&\left(\mu_{2_F} - 1\right) d\tilde{h}_F = 0 \ , \quad d\tilde{h}_F \neq 0 \quad \therefore \ \mu_{2_F} = 1 \\
&C \ d\tau_F = 0 \ , \quad d\tau_F \neq 0 \quad \therefore \ C = 0 = const.
\end{aligned} \tag{51}$$

Consequently, it follows that, at the terminal point F

$$\mu_{3_F} = \left[\frac{z}{\tilde{m}}\left(1 - \frac{\tilde{D}}{\tilde{\beta}}\right)\right]_F \tag{52}$$

If the final time τ_F were given, then $C = const. \neq 0$ and

$$\mu_{3_F} = \left[\frac{1}{\tilde{\beta}}\left(C - \frac{z\tilde{D}}{\tilde{m}}\right) + \frac{z}{\tilde{m}}\right]_F \ . \tag{53}$$

Eqs. (52) and (53) indicate that no arrival at the end-point F can be performed by a coasting sub-arc $(\tilde{\beta} = 0)$ otherwise $\mu_{3_F} = \infty$.

4. THE AERODYNAMIC DRAG FUNCTION AND CONDITIONS ALONG THE β -VAR. SUB-ARC

In the analysis developed in previous paragraphs an aerodynamic drag function of the general functional form $\tilde{D} = \tilde{D}(z, \tilde{h})$, has been assumed. On this basis a generalized expression for the optimum burning program has been derived. The aerodynamic drag function will now be given two specific forms in order to analyze particular applications of previous results. These forms will be used later for the numerical solution of some examples. In this analysis we will refer to free-time problems $(C = 0)$.

For the sake of discussion arbitrary aerodynamic drag functions of the forms

$$
\begin{aligned}
&a) \quad \tilde{D} = k_1 z^2 = \tilde{D}(z) \;, \quad k_1 = \text{const.} \\
&b) \quad \tilde{D} = k_1 z^2 e^{-k_2 \tilde{h}} = \tilde{D}(z, \tilde{h}) \;, \quad k_1, k_2 = \text{Const.}
\end{aligned}
\tag{54}
$$

will be considered. These expressions, however, have some engineering meaning. For example, the form (54a) corresponds to the case of flight in an atmosphere of constant density and has been applied in previous investigations (i.e., Refs. 6, 7, 11). The form in (54b) corresponds to assuming the hypothetical case of a rocket having constant zero-lift drag coefficient and flying in an exponential atmosphere.

4.1 Case $\tilde{D} = k_1 z^2 = \tilde{D}(z)$

In this case, Eq. (38) leads to

$$\tilde{m} = \tilde{D}(1+z) = \tilde{m}(z) \tag{55}$$

From Eq. (39), the optimum burning program or control program is

$$\tilde{\beta} = \tilde{D}\,\frac{4 + z(8+3z)}{2 + z(4+z)} = \tilde{\beta}(z) \tag{56}$$

Finally, from Eqs. (1),(55) and (56), it may be derived that the acceleration along the $\tilde{\beta}$ -var. sub-arc is given by

$$z' = -\frac{z^3 + 3z^2 + 2z}{z^3 + 5z^2 + 6z + 2} = z'(z) \tag{57}$$

Eq. (57) shows that along the $\tilde{\beta}$ -var. sub-arc the vehicle decelerates, which is consistent with what is implied in Eq. (55) since $\tilde{m}$ must decrease. .

In this case the numerical solution of practical applications is simple due to the two-dimensional character of the closed-form expressions (55) to (57). By the same token, and as will be shown later in the examples, the determination of corner points is straight forward.

4.2 Case $\tilde{D} = k_1 z^2 e^{-k_2 \tilde{h}} = \tilde{D}(z,\tilde{h})$

From Eq. (38) is now obtained

$$\tilde{m} = \tilde{D}(1+z) = \tilde{m}(z,\tilde{h}) \tag{58}$$

The optimum burning program or control program is obtained from Eq. (39) as

$$\tilde{\beta} = \tilde{D} \, \frac{(2+3z)(2+z) + k_2 z^2 (1+z)^2}{2 + z(4+z)} = \tilde{\beta}(z, \tilde{h}) \tag{59}$$

The optimum acceleration along the $\tilde{\beta}$ -var. sub-arc is obtained from Eqs. (1), (58) and (59) as

$$z' = \frac{k_2 z^4 + (2k_2 - 1) z^3 + (k_2 - 3) z^2 - 2z}{[2 + z(4+z)](1+z)} =$$

$$= \frac{k_2 z^3 + (k_2 - 1) z^2 - 2z}{z^2 + 4z + 2} = z'(z) \tag{60}$$

In this case, the solution of numerical applications is more complex than in the previous case due to the tri-dimensional character of Eqs. (58) and (59). It may be readily seen that for $k_2 = 0$, Eqs. (58), (59) and (60) reduce to Eqs. (55), (56) and (57) respectively. As will be shown in the examples, the determination of corner points can now be done using the corner line. This technique will be discussed in more detail later. Along the $\tilde{\beta}$ -var. sub-arc the equations of motion and Eq. (59) require the use of numerical methods of integration.

5. EXAMPLES - MAXIMUM ALTITUDE FOR GIVEN FUEL CONSUMPTION AND FREE FINAL TIME

The object of this paragraph is to show the application of the previous theory to the numerical solution of a given boundary-value problem. In particular, for the case where $\tilde{D} = \tilde{D}(z, \tilde{h})$. Special emphasis is placed on the determination of corner points and on the integration of the admissible set of variable Lagrange multipliers, or adjoint variables.

For the problem proposed, the following boundary conditions will be assumed:

$$\begin{aligned} \psi_1 &\equiv z_I = 0 & \psi_4 &\equiv \tau_I = 0 \\ \psi_2 &\equiv \tilde{h}_I = 0 & \psi_5 &\equiv z_F = 0 \\ \psi_3 &\equiv \tilde{m}_I - 1 = 0 & \psi_6 &\equiv \tilde{m}_F - 0.4 = 0 \end{aligned} \tag{61}$$

The functional to be extremized is written

$$G = \tilde{h}_I - \tilde{h}_F = -\tilde{h}_F = min. \tag{62}$$

As indicated in sub-paragraph 3.1, for this problem the following natural boundary conditions are obtained

$$\mu_{2_F} = 1 \quad , \quad C = 0 = const. \tag{63}$$

For the first example, the following values will be taken;

$$\begin{aligned} \tilde{D} &= k_1 z^2 \exp(-k_2 \tilde{h}) \quad , \quad k_1 = 5 \quad , \quad k_2 = 11.87 \\ \tilde{\beta}_{max.} &= 2 \quad , \quad \tilde{\beta}_{min.} = 0 \end{aligned} \tag{64}$$

As indicated in paragraph 1, [Eq. (13)], three types of sub-arcs are admissible; $\tilde{\beta}_{max.}$, $\tilde{\beta} - var.$ and $\tilde{\beta} = \tilde{\beta}_{min.} = 0$. The sequence of these sub-arcs will be discussed in the following. Due to the launching conditions in (61), obviously no $\tilde{\beta} = 0$ sub-arc may be started at the initial point I. Also, if the initial values in (61) are replaced in the function (35), we get

$$\varphi_I = \varphi(z_I, \tilde{h}_I, \tilde{m}_I) = 1 \tag{65}$$

from which we concluded that no $\tilde{\beta} - var.$ sub-arc may be started at I. In fact, this also can be readily verified from Eq. (60) which gives the acceleration z' along the $\tilde{\beta} - var.$ sub-arc for different values of the parameter k_2. This is shown in Fig. 2.

Thus, at point I the only sub-arc that may be started is $\tilde{\beta}_{max.} = 2 = const.$ (see Fig. 3). Starting at I with a $\tilde{\beta}_{max.}$ sub-arc, our next problem is how to determine the position of an eventual corner point. For that, we will make use of the corner line. Integrating numerically the set of equations (1) to (3) with $\tilde{\beta} = \tilde{\beta}_{max.}$, the state variables $z(\tau)$, $\tilde{h}(\tau)$, $\tilde{m}(\tau)$ are obtained. The integration was performed using an I. B. M. 1620 computer and applying the Runge-Kutta step-integration method. Replacing the functions $z(\tau)$ and $\tilde{h}(\tau)$ obtained along the $\tilde{\beta}_{max.}$ sub-arc in the equation

$$\tilde{m} = z(\tilde{D}_z + \tilde{D}) - \tilde{D} = \tilde{m}[z(\tau), \tilde{h}(\tau)] \tag{66}$$

the resulting function $\tilde{m}(\tau)$, may now be plotted in the $(\tilde{m}, z)$ -plane. The line so obtained in the $(\tilde{m}, z)$ -plane, will be called corner line (this line must not be confused with loci of corners), and is shown in Fig. 4 identified by the letter ℓ . Since at corners the state variables $z(\tau)$, $\tilde{h}(\tau)$ and $\tilde{m}(\tau)$ are assumed all continuous (see Fig. 5), a corner point may only exist at the intersection of the corner line ℓ and the β_{max} sub-arc. Several examples are shown in Fig. 4, in which boosters delivering different thrust levels, i. e., $\tilde{\beta}_{max} = \infty$, 5, 2 and 1.5 have been assumed. The line $A_1 A_2 A A_3$ in Fig. 4 is the loci of corners mentioned before. In the particular case $\tilde{\beta}_{max} = 2$ here considered, the corner point is identified in Fig. 4 with the letter A . The corner line ℓ was extended past the point A in order to verify the non-existence of other possible corners. The line ℓ , within the range of interest,is monatomically increasing and therefore only one corner point appears admissible. In the numerical calculation of the corner point A , an automatic stopping condition was included in the computer program as

$$\left| 1 - \tilde{\beta}_{max} \tau - k_1 z^2 exp\left(-k_2 \tilde{h}\right) \cdot \left(z + 1\right) \right| \leqq \epsilon \tag{67}$$

where $\epsilon > 0$ is the precision with which the corner point A (Fig. 3) is desired and $z(\tau)$, $\tilde{h}(\tau)$ are the values along the $\tilde{\beta}_{max} = 2$ sub-arc. At the point A , the transition to the $\tilde{\beta}-var.$ sub-arc was made, as shown in Fig. 3. Along the $\tilde{\beta}-var.$ sub-arc, the set of Eqs. (1) to (3) with $\tilde{\beta}$ replaced from Eqs. (39) or (59) was integrated up to the point B (Fig. 3) where $\tilde{m} = \tilde{m}_F = 0.4$. No departure from the $\tilde{\beta}-var.$ sub-arc may exist between A and B, since this sub-arc is interior to the region of admissible displacement determined by the combination $\tilde{\beta}_{max.} \rightarrow \tilde{\beta} = 0$. This combination satisfies the prescribed boundary conditions and within this region, both-sided control variations $\delta\tilde{\beta} \gtreqless 0$ leading to $H_{\tilde{\beta}} = 0$, $(i.e., \tilde{\beta}\text{-}var.)$, are admissible. From point B the vehicle coasts to F (Fig. 3) to meet the boundary condition $z_F = 0$. It is easy to see from Fig. 3 that depending on the final values of z_F and/or $\tilde{m}_F$ either one of the last two sub-arcs or both of them may not exist.

A graphical representation of the extremal arc in the space of state variables $(z, \tilde{h}, \tilde{m})$ is shown in Fig. 5. In order to verify the extremal properties of the arc $IABF$, position of the corner points and sequence of sub-arcs, the adjoint variables or Lagrange multipliers may be integrated backwards. Since at the final point

$$\left[\mu_1 \left(\frac{\tilde{D}}{\tilde{m}} + 1\right)\right]_F = \mu_{2_F}\, z_F \tag{68}$$

and $z_F = 0$, then it follows that $\mu_{1_F} = 0$. Now, Eqs. (14) and (15) may be integrated backwards $(F \rightarrow B)$, yielding $\mu_1(\tau)$ and $\mu_2(\tau)$ along the $\tilde{\beta} = 0$ sub-arc (FB sub-arc in Fig. 3). At the point B, the multiplier μ_{3_B} is found using Eqs. (13a) and (18). Now, Eq. (16) may be integrated forward again to calculate

$$\mu_3(\tau) = \int_{\tau_B}^{\tau} \frac{\mu_1}{\tilde{m}^2} \left(\tilde{\beta} - \tilde{D}\right) d\tau + \mu_{3_B} \tag{69}$$

along the $\tilde{\beta} = 0$ sub-arc. Backwards integration of the system of Eqs. (14), (15) and (16), from the point B, permits the calculation of all the adjoint variables associated with the extremal $IABF$. The result of the numerical integration performed for the present example is shown in Fig. 6, where the multipliers $\mu_1(\tau)$, $\mu_2(\tau)$ and $\mu_3(\tau)$ are shown.

Note that at the corners A and B, the function $\mu_3'(\tau)$ is discontinuous due to the discontinuity $\Delta\tilde{\beta}$ in the control variable. Finally, the function $H_{\tilde{\beta}}(\tau)$ may be calculated along the extremal and is shown in Fig. 7. Note that, as indicated before, for $H_{\tilde{\beta}} \geqq 0$, $\tilde{\beta} = \tilde{\beta}_{max.}$; for $H_{\tilde{\beta}} = 0$, $\tilde{\beta} = var.$ and for $H_{\tilde{\beta}} \leqq 0$, $\tilde{\beta} = \tilde{\beta}_{min.}$. At corners A and B, $H_{\tilde{\beta}} = 0$.

Finally, the same boundary-value variational problem was solved using a drag function $\tilde{D} = k_1 z^2$, $k_1 = 5 = const.$ In this case there was no problem in the determination of the corner point A since the $\tilde{\beta}$-$var.$ sub-arc in the $(\tilde{m}, z)$ -plane may be drawn a priori. In fact, from Eq. (55)

$$\tilde{m} = 5 z^2 (1 + z) \tag{70}$$

along the $\tilde{\beta}$-$var.$ sub-arc. The solution of the problem in this case is shown in Fig. 8.

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REFERENCES

1. Bliss, G.A., "Lectures on the Calculus of Variations", The University of Chicago Press, Chicago, 1946.

2. Carathéodory, C., "Variationsrechnung und Partielle Differential gleichungen erster Ordnung", 1935.

3. Bolza, O., "Lectures on the Calculus of Variations", Stechert-Hafner, Inc., New York, 1946.

4. Cicala, P., "An Engineering Approach to the Calculus of Variations", Libreria Editrice Universitaria Levrotto & Bella, Torino, Italy, 1957.

5. Cavoti, C.R., "The Calculus of Variations Approach to Control Optimization", Special Report No. 1, Contract NAS 8-2600, Prepared for George C. Marshall Space Flight Center, NASA, Huntsville, Alabama, June 15, 1962.

6. Leitmann, G., "An Elementary Derivation of the Optimal Control Conditions", XIIth IAF Congress, Washington, D.C., October 2-7, 1961.

7. Berkovitz, L. D., "An Optimum Thrust Control Problem", Journal of Mathematical Analysis and Applications. 3, 122-132, 1961.

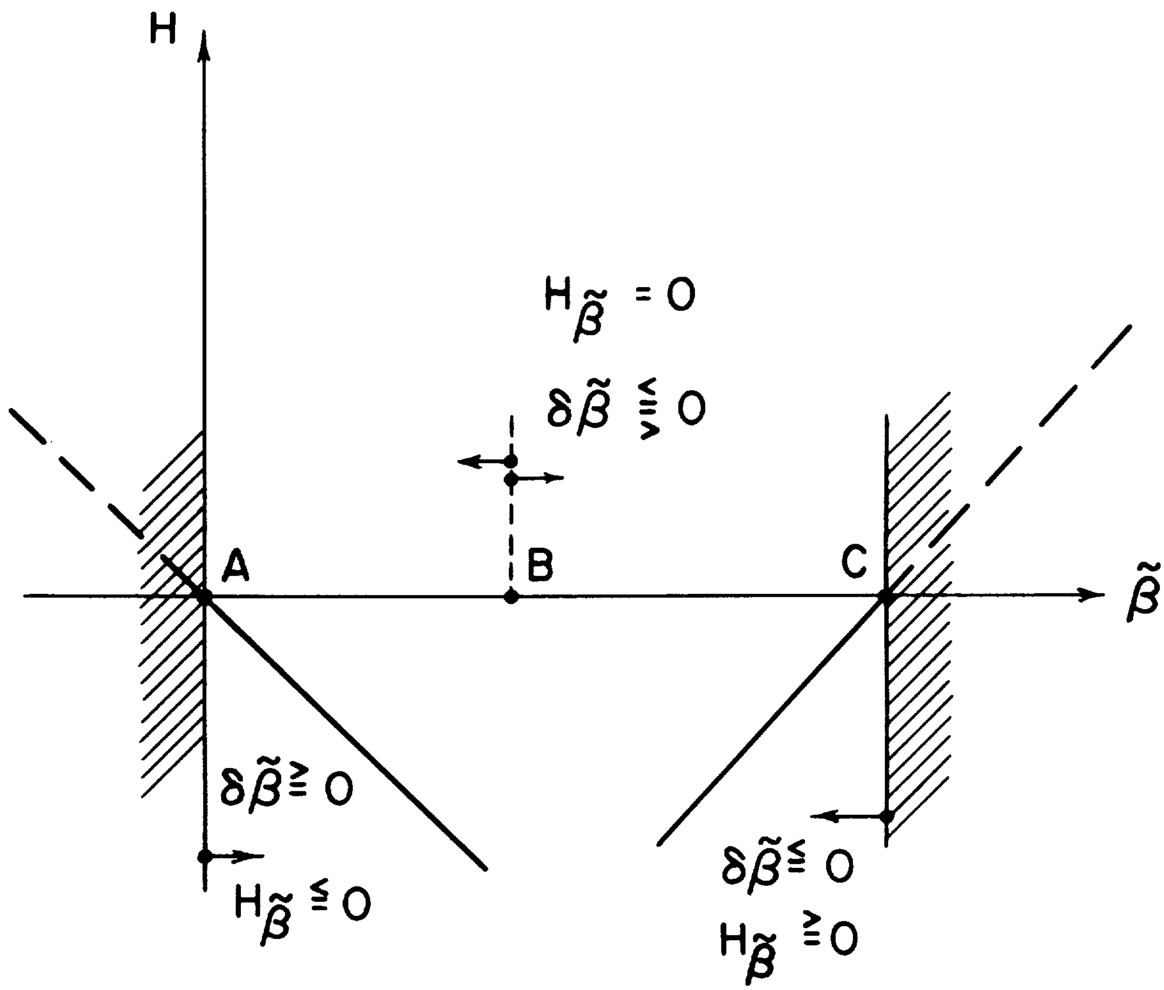
8. Miele, A., and Cavoti, C.R., "Optimum Thrust Programming along Arbitrarily Inclined Rectilinear Paths", Astronautica Acta, Vol. IV, Fasc. 3, 1958.

9. Leitmann, G., "Stationary Trajectories for a High-Altitude Rocket with Drop-Away Booster", Astronautica Acta 2, 119, 1956.

10. Boltyanskiĭ, V.G., Gamkrelidze, R.V., Pontryagin, L.S., "The Theory of Optimal Process. I, The Maximum Principle", American Mathematical Society Translation, Series 2, Vol. 18, 1961.

11. Miele, A., "Generalized Variational Approach to the Optimum Thrust Programming for the Vertical Flight of a Rocket. Part I, Necessary Conditions for the Extremum", Z.F.W., Vol. 3, March 1958.

LIST OF CAPTIONS


H
$H_{\tilde{\beta}} = 0$
$\delta\tilde{\beta} \lesseqqgtr 0$
A
B
C
$\tilde{\beta}$
$\delta\tilde{\beta} \geqq 0$
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$\delta\tilde{\beta} \leqq 0$
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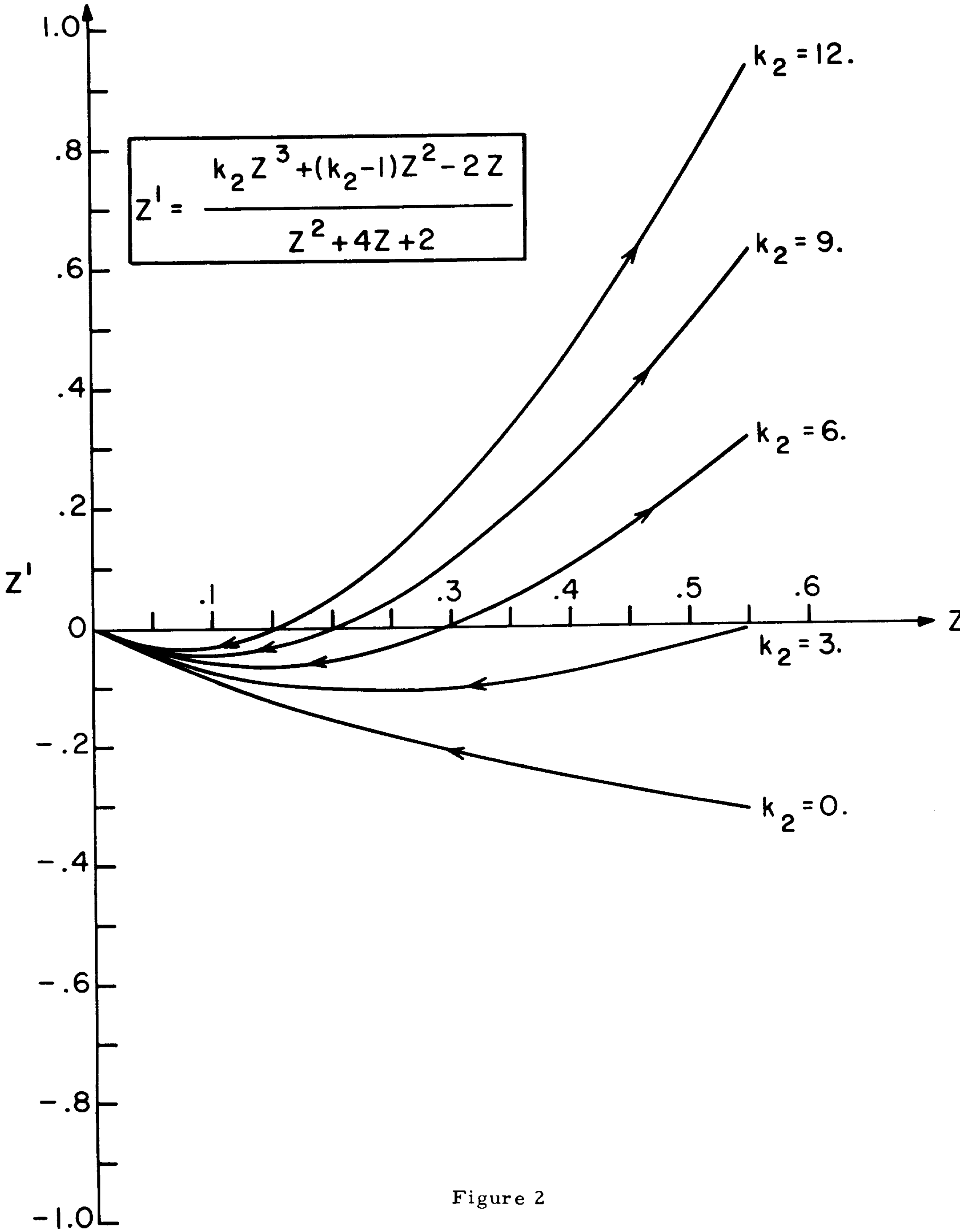
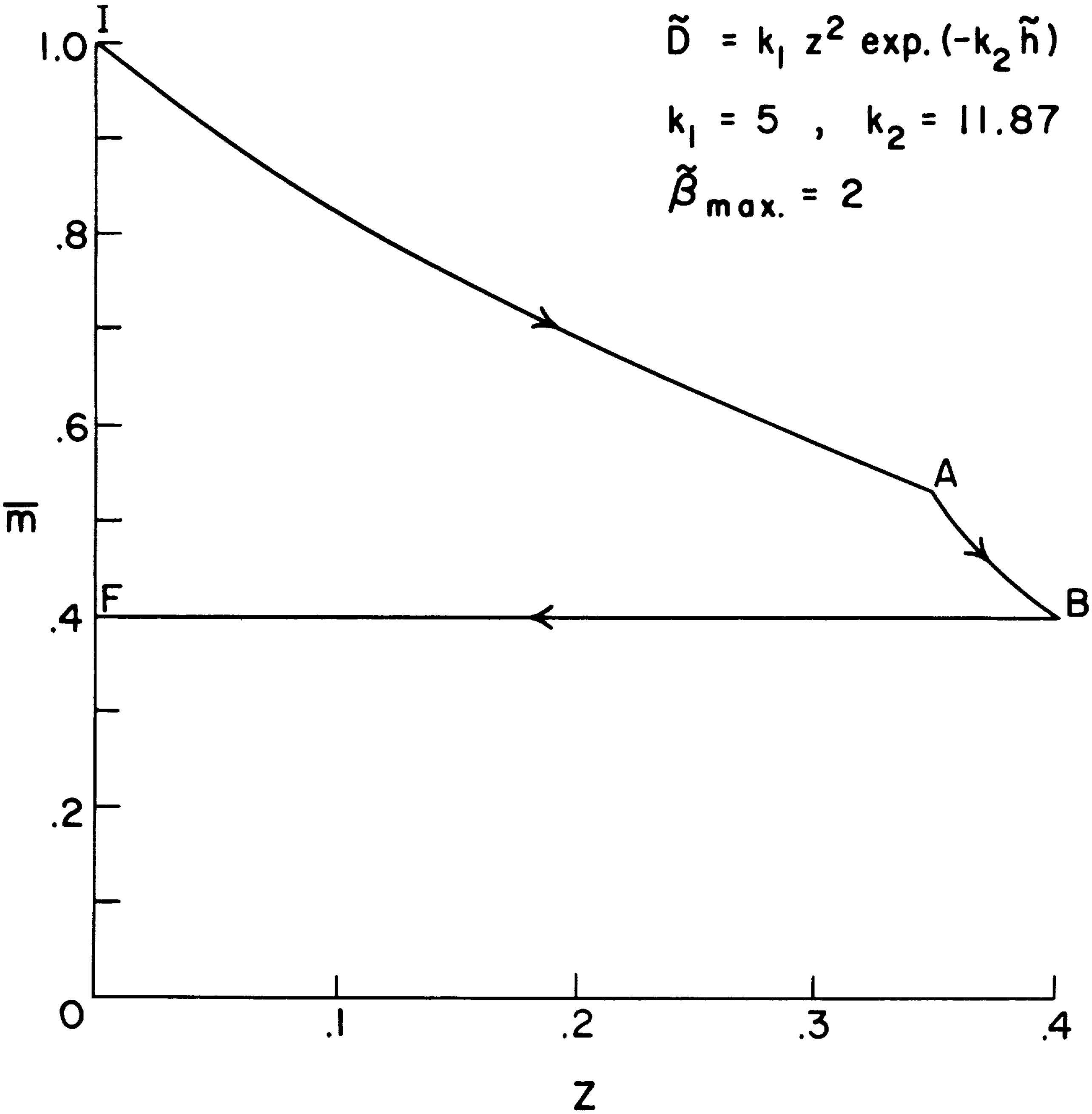
Figure 1



1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
Z'
.1
.3
.4
.5
.6
Z
$Z' = \dfrac{k_2 Z^3 + (k_2 - 1)Z^2 - 2Z}{Z^2 + 4Z + 2}$
$k_2 = 12.$
$k_2 = 9.$
$k_2 = 6.$
$k_2 = 3.$
$k_2 = 0.$

Figure 2



$\tilde{D} = k_1 z^2 \exp.(-k_2 \tilde{h})$
$k_1 = 5$, $k_2 = 11.87$
$\tilde{\beta}_{max.} = 2$
I
A
B
F
1.0
.8
.6
.4
.2
0
.1
.2
.3
.4
$\overline{m}$
Z

Figure 3

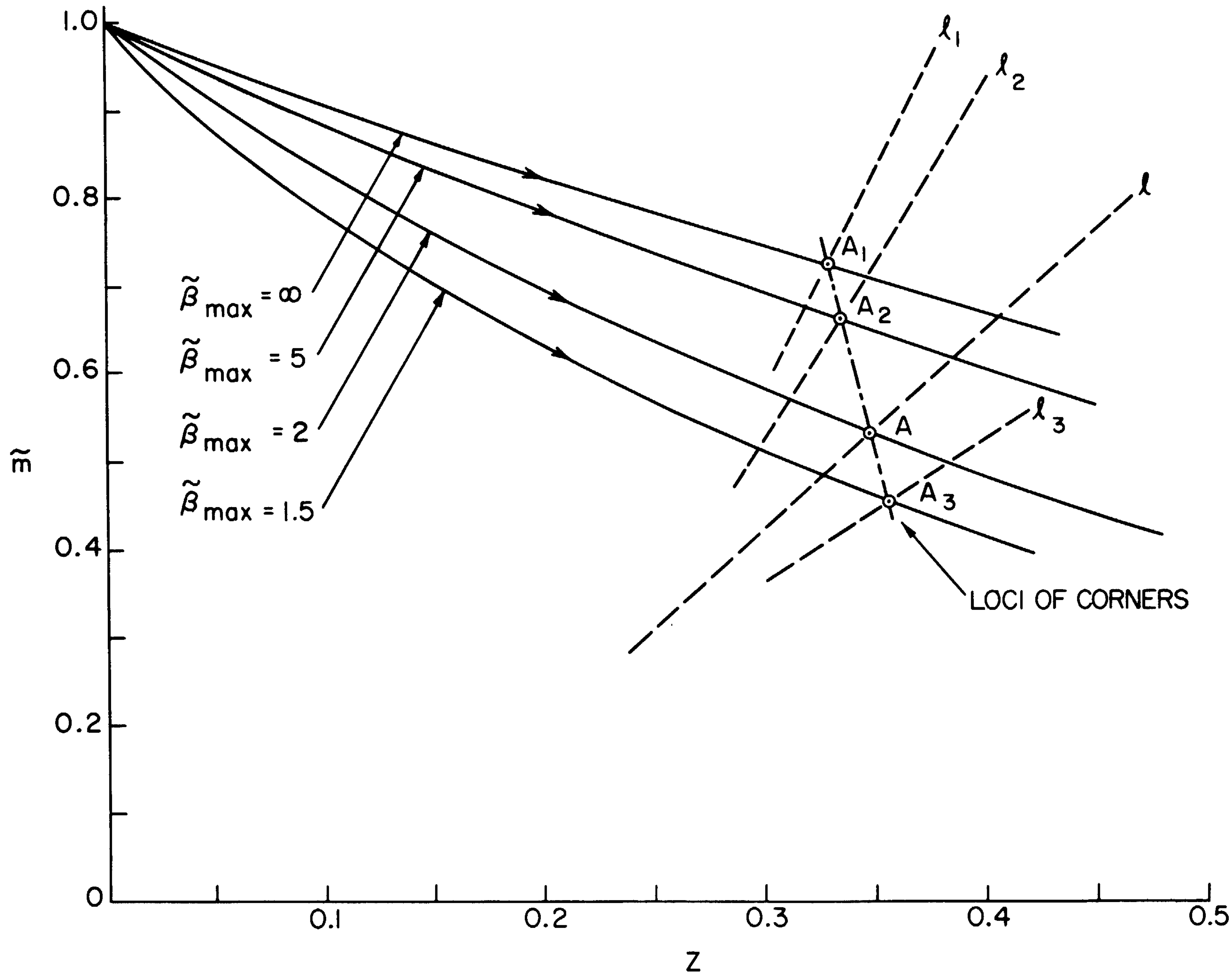

1.0
0.8
0.6
0.4
0.2
0
$\tilde{m}$
0.1
0.2
0.3
0.4
0.5
Z
$\tilde{\beta}_{max} = \infty$
$\tilde{\beta}_{max} = 5$
$\tilde{\beta}_{max} = 2$
$\tilde{\beta}_{max} = 1.5$
ℓ_1
ℓ_2
ℓ
ℓ_3
A_1
A_2
A
A_3
LOCI OF CORNERS

Figure 4

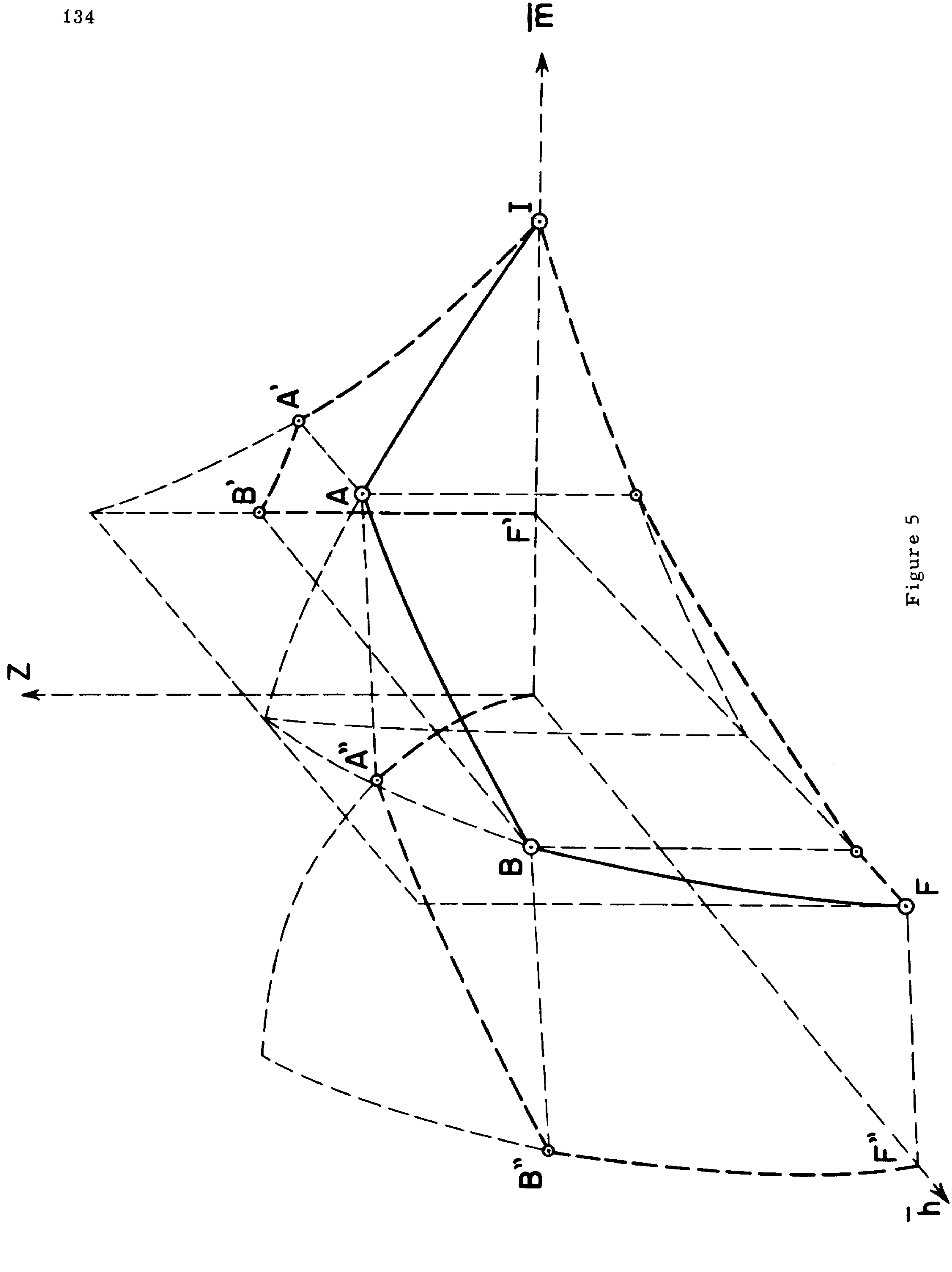


Figure 5

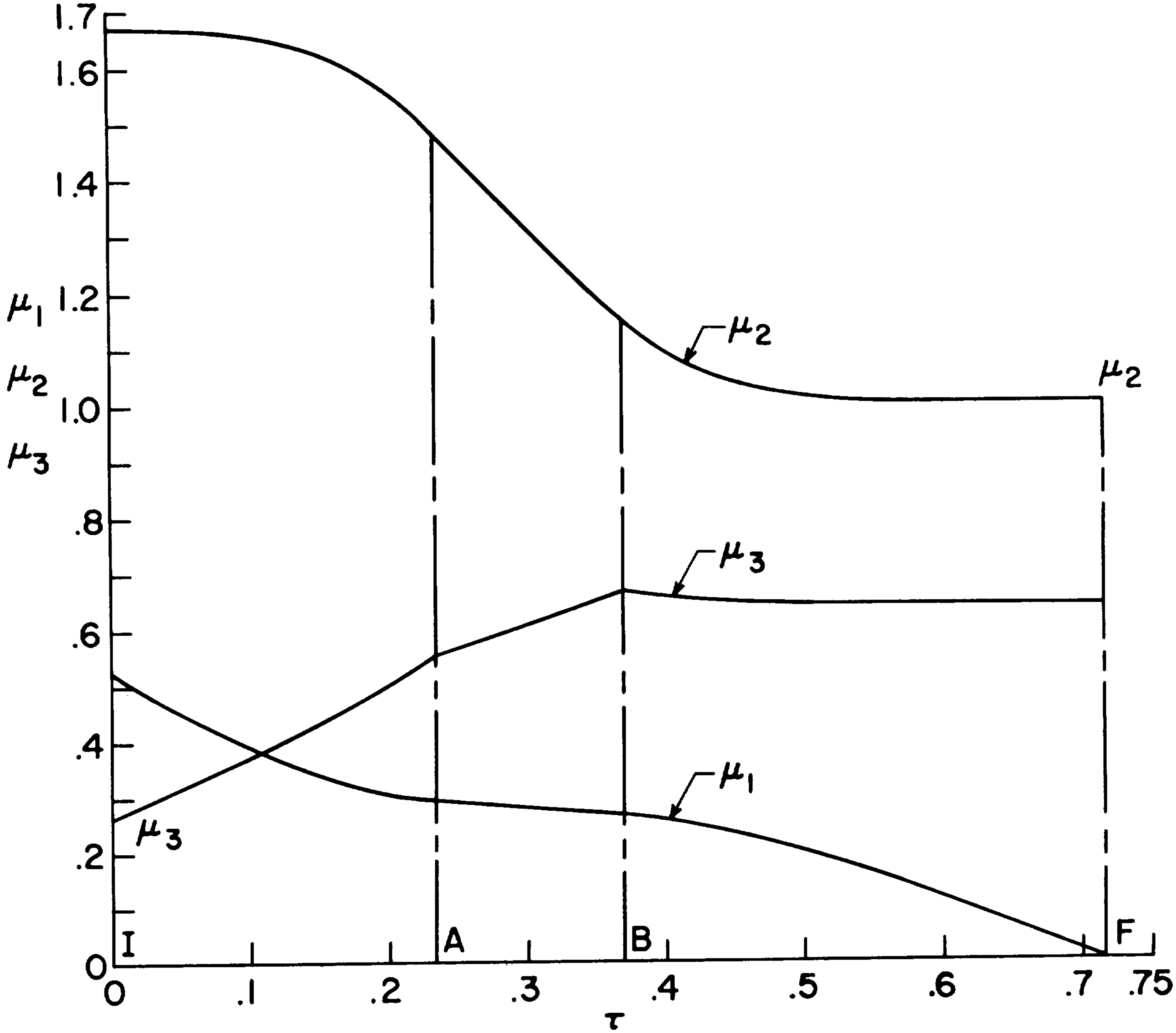


Figure 6

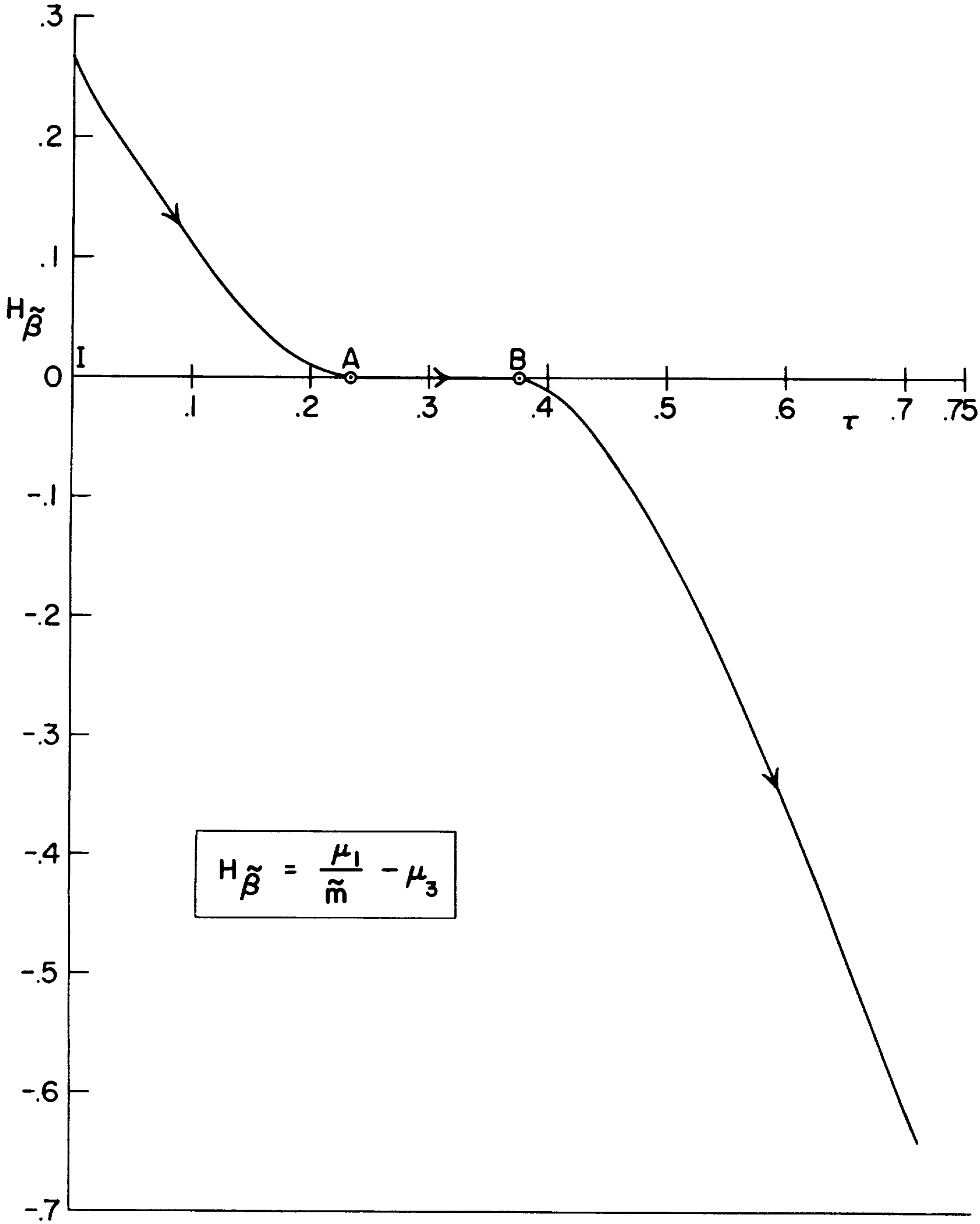
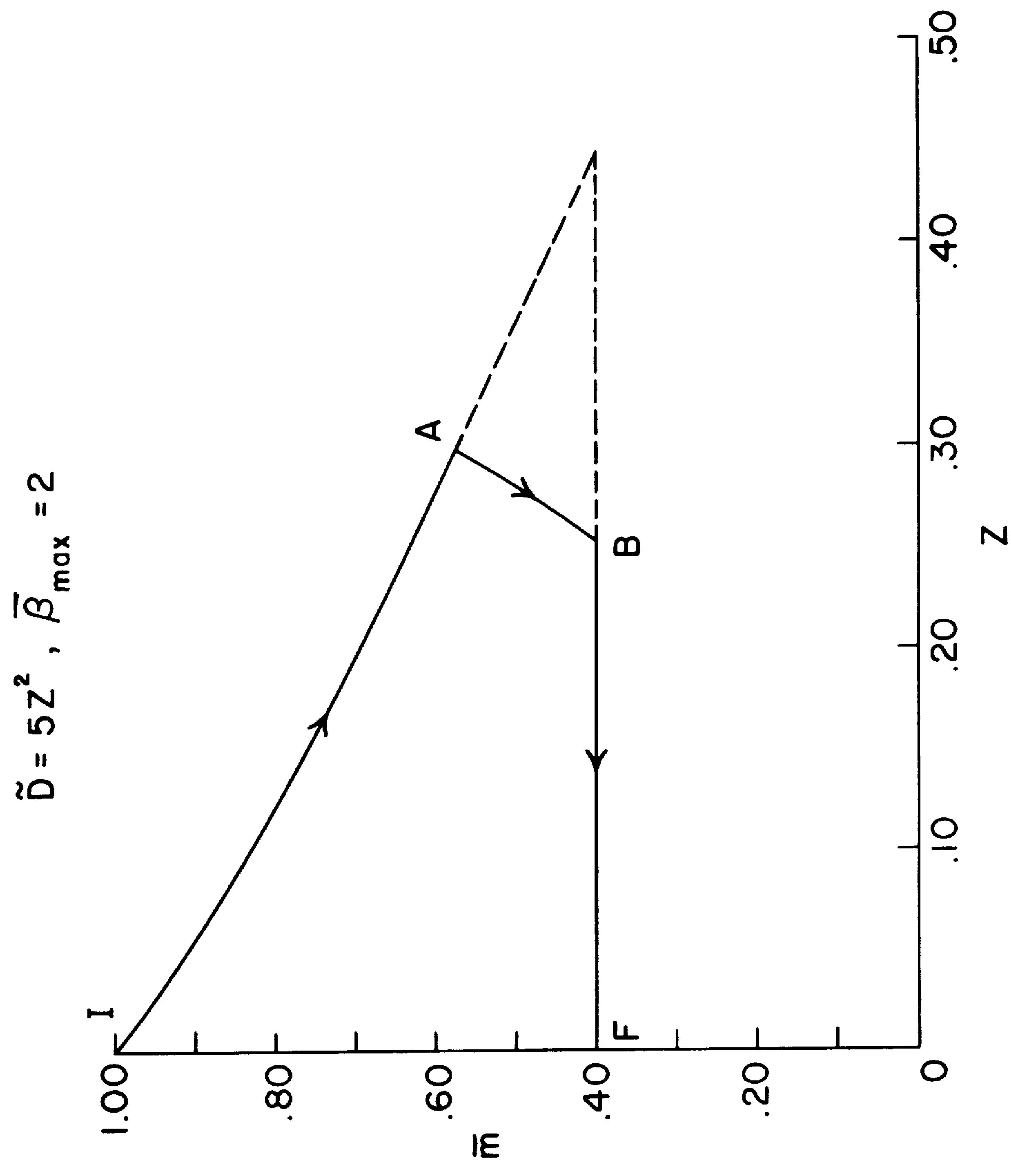

.3
.2
.1
$H_{\tilde{\beta}}$
I
A
B
0
.1
.2
.3
.4
.5
.6
τ
.7
.75
-.1
-.2
-.3
-.4
$H_{\tilde{\beta}} = \frac{\mu_1}{\tilde{m}} - \mu_3$
-.5
-.6
-.7

Figure 7



$\tilde{D} = 5Z^2$, $\bar{\beta}_{max} = 2$
A
B
I
F
1.00
.80
.60
.40
.20
0
$\bar{m}$
.10
.20
.30
.40
.50
Z

Figure 8

RENDEZVOUS POSSIBILITIES WITH THE IMPULSE OF OPTIMUM TWO-IMPULSE TRANSFER

by

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National Aeronautics and Space Administration
Huntsville, Alabama

NOTATION

e, i, p, P, ω	Elements of elliptical orbits: eccentricity, inclination, semi-latus rectum, period, perigee angle, respectively. Subscripts 1, 2, 3 refer to initial, target, and transfer orbits, respectively.
t_1, t_2, t_3	Time intervals on orbits corresponding to ϕ_1, ϕ_2, and $\Delta\theta$, respectively
τ	Relative time of nodal passage (positive when target passes before ferry)
$\Delta\theta$	Angle between departure and arrival on transfer orbit
$\underline{\phi} = (\phi_1, \phi_2, \phi_3)$	
ϕ_1	Angle between departure point and node or reference
ϕ_2	Angle between arrival point and node or reference
ϕ_3	True anomaly of arrival point in transfer orbit

RENDEZVOUS POSSIBILITIES WITH THE IMPULSE OF OPTIMUM TWO-IMPULSE TRANSFER

By

D. F. Bender

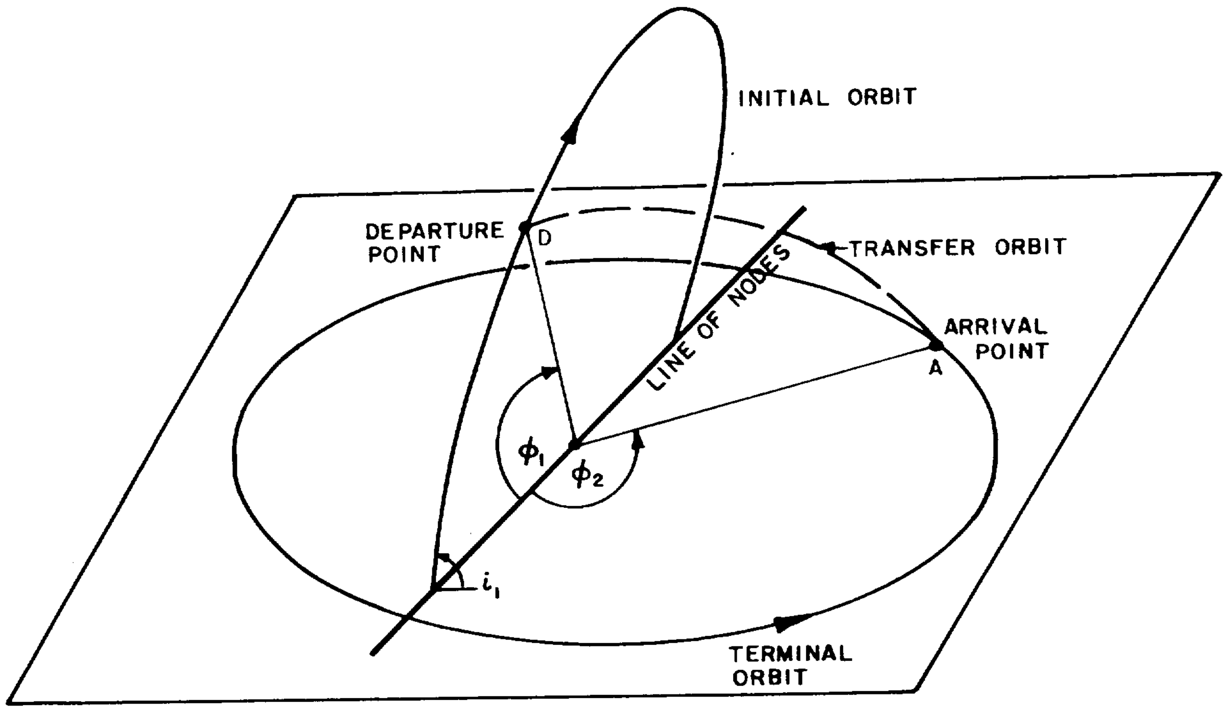
16807

Summary

Two-impulse optimum orbital transfer will lead to rendezvous for only a single value of the relative positions of the ferry and target at the beginning of the maneuver. It will be shown that, by the simple expedient of splitting either the first or second impulse of this optimum transfer between any two orbits and holding for one revolution in the intermediate transfer orbit so obtained, rendezvous is possible over an extended range of relative phases. A technique for generating the required data including excess thrust tolerances for any orbit pair will be explained and sample data presented.

I. INTRODUCTION

In a large number of cases optimized two-impulse transfer between two orbits around an attracting center requires less total impulse than any other type. If a rendezvous at the conclusion of such an orbit change is required, then it is clear that in general only one value of the relative positions of the two objects at the beginning will be allowable. It is desirable to discover the widths of such minima so



PROJECTION ON UNIT SPHERE

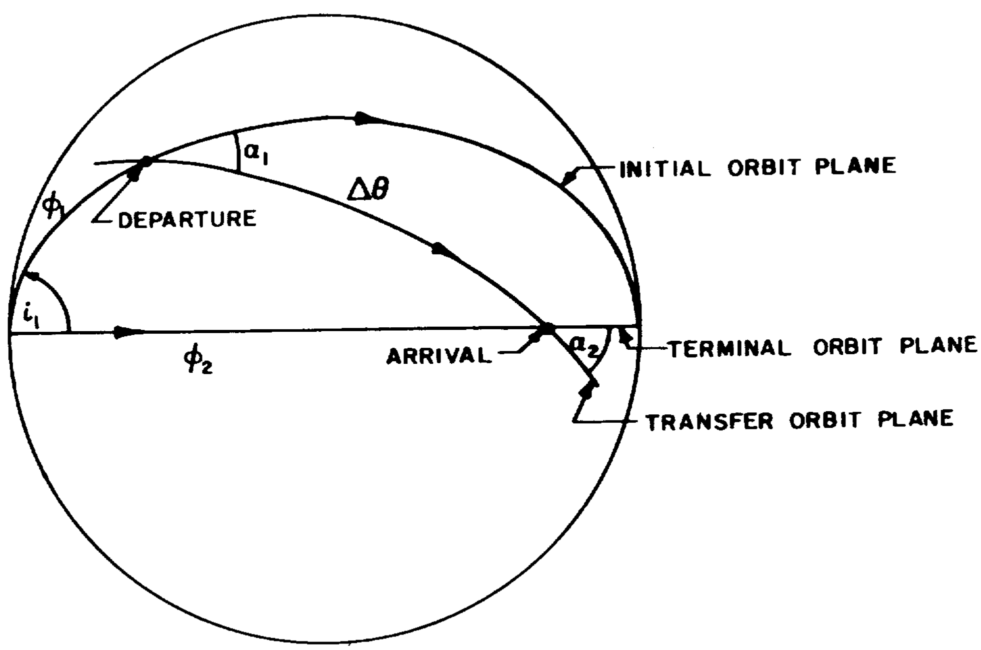


FIG. 1. TRANSFER GEOMETRY

that error sensitivities can be indicated. In addition, it will be shown that by permitting three impulses the minimum of the total impulse versus relative phase for any pair of orbits can be made to have a horizontal base whose width is at least the difference in periods of the initial and final orbits. The technique is to split one of the two impulses into two parts used exactly one revolution apart.

The geometry of the transfer is indicated in Figure 1, in which the orbit planes are projected onto a unit sphere. If the orbit planes of the passive target satellite (terminal or 2) and the ferry vehicle (initial or 1) are inclined as indicated, the line of nodes is taken as the reference direction with the ferry ascending. If not, one uses any direction in the plane, usually the perigee of one of the orbits. The relative phase of the two is given by indicating the position of the target (2) when the ferry (1) crosses the reference direction line. In this discussion the time interval τ is used, that is, the target is at a position corresponding to the time τ past the reference line when the ferry crosses it.

Two-impulse transfer has been extensively studied as a perusal of the aerospace and astronautics journals will indicate. For this discussion it is necessary to have a computer program which is able to survey and optimize on total impulse so as to select from all the possible transfers those requiring the least fuel. Such a program has been developed by Kerfoot and DesJardins[1] and further improved by McCue[2] of the Space Sciences Laboratory of S&ID, NAA.

The formulation of the two-impulse transfer problem by DesJardins and Kerfoot is to express the total impulse as a function of three angles (ϕ_1, ϕ_2, ϕ_3), or (ϕ). The angle ϕ_3 is the variable which selects a particular transfer orbit between the departure point (ϕ_1) and the arrival point (ϕ_2). The variable used is the true anomaly of the arrival point in the transfer orbit considering $\Delta\theta < \pi$. In all cases both a short ($\Delta\theta < \pi$) and a long ($\Delta\theta > \pi$) transfer are considered and the better one selected. (The discarded transfer corresponds to motion in the opposite sense on the transfer orbit.)

The full range of phasing possibilities is encompassed by $0 \leq \tau \leq P_2$ (where P_2 is the period in the second orbit), since any value of τ outside this range may be treated modulo P_2. Consider next the ensuing revolution of the ferry. The value of τ has decreased by the difference $P_2 - P_1$, and it is clear that on successive orbits of the ferry the value of τ will continue to step by this difference.

If t_1, t_2, and t_3 are the traverse times associated with the true anomaly intervals ϕ_1, ϕ_2, and $\Delta\theta$ then a necessary and sufficient condition for rendezvous is $t_1 + t_3 = t_2 - \tau$ or $\tau = t_2 - t_1 - t_3$. Since t_1, t_2, and t_3 are functions of $\underline{\phi}$, it follows that τ is also a function of $\underline{\phi}$. Thus, finding optimum two-impulse rendezvous trajectories is a matter of minimizing $I(\underline{\phi})$ under the constraint $\gamma(\underline{\phi})$ = constant. It is easy to see that this consists of finding points in ϕ space at which the surfaces I = constant and τ = constant are parallel. Analytical expressions for all the required derivatives can be obtained easily[3] but are not given here. They were programmed and used in the searching technique for the time constrained optima.

II. THE IMPULSE SPLITTING TECHNIQUE

The relative phase, τ, between two vehicles changes by the difference in periods each time the initial vehicle crosses the reference axis. If a vehicle holds in the initial orbit, the decrease in τ can only be $P_2 - P_1$. If, on the other hand, a hold is possible in some intermediate orbit of period P', the effective value of τ will decrease by $P_2 - P'$. By splitting either I_1 or I_2 into two parts, the second of which is used one revolution after the first, any period P' between P_1 and P_2 may be attained. Thus any value of τ lying between an optimum τ_o and $\tau_o + (P_2 - P_1)$ is effectively reduced to τ_o and made accessible for rendezvous. This technique is similar to the looping methods described by Silber and Hoelker.[4] Two immediate consequences are evident:

(1) An upper bound is provided on the number of waiting periods in the initial orbit before an optimum rendezvous maneuver may be performed. The bound, n_{max}, is given by

$$n_{max} \leq \frac{P_2}{P_2 - P_1}$$

(2) If as many as $n_{max} + 1$ revolutions are permissible, it is then true that a three-impulse rendezvous maneuver can be made which requires no more fuel than the optimum two-impulse orbital transfer.

For the case when $P_1 \leq P_3 \leq P_2$, P' must lie in the range P_1 to P_2, and thus the curves of constrained optima are translated from the positions indicated in Figures 3 and 6 to points $(P_2 - P_1)$ greater and the minimum value of impulse may be achieved anywhere in the interval. If the transfer orbit period happens to lie outside the range P_1 to P_2, greater ranges of P' and hence of τ are accessible. If $P_3 < P_1$ the range for τ extends to $P_2 - P_3$ above that of the constrained optimum. If $P_3 > P_2$, the range is from $P_3 - P_2$ below to $P_2 - P_1$ above. Both of these cases occur in Figure 6.

Properties of the intermediate transfer orbit depend upon the geometry of the transfer problem and the fraction of impulse used in the phasing maneuver. Velocity components may be determined; then angular momentum and energy may be obtained and used to calculate the elements of the intermediate orbit.

Up to this point discussion has been limited to multiple holds in the initial orbit. In any case where P_3 lies outside the range P_1 to P_2, fewer total holds may become possible if more than one is taken in an intermediate transfer orbit.

III. TYPICAL NUMERICAL RESULTS

Numerical results using this technique are shown for two different pairs of orbits in two sets of figures: Figures 2, 3, and 4 for a pair of circles inclined at 3°5 and Figures 5, 6, and 8 for a pair of inclined and asymmetrically oriented ellipses.

The two circular orbits of Figures 2, 3, and 4 have radii of 4070 miles and 4270 miles respectively. The optimum impulse versus departure point is shown in Figure 2 where the effects of the inclination are evident. Since there is no orbital distinction between the two nodes, only one optimum needs to be explored in the search for impulse versus relative time (τ), Figure 3. The stepping of τ can be considered to be with steps of $(P_2 - P_1)/2$ every half revolution of the ferry.

These two circular orbits are close together in period and illustrate the need for proper phasing if rendezvous is to be accomplished in a few revolutions, since even the extention of the minimum of Figure 3 by $P_2 - P_1$ to the left covers only a small fraction of the total period, P_2. In Figure 4 the impulse splitting possibilities are illustrated for the optimum and for three points on the curve of Figure 3 requiring slightly greater total impulse. The excess is indicated as a percentage on each line. If the line slopes upward to the left the first impulse is to be

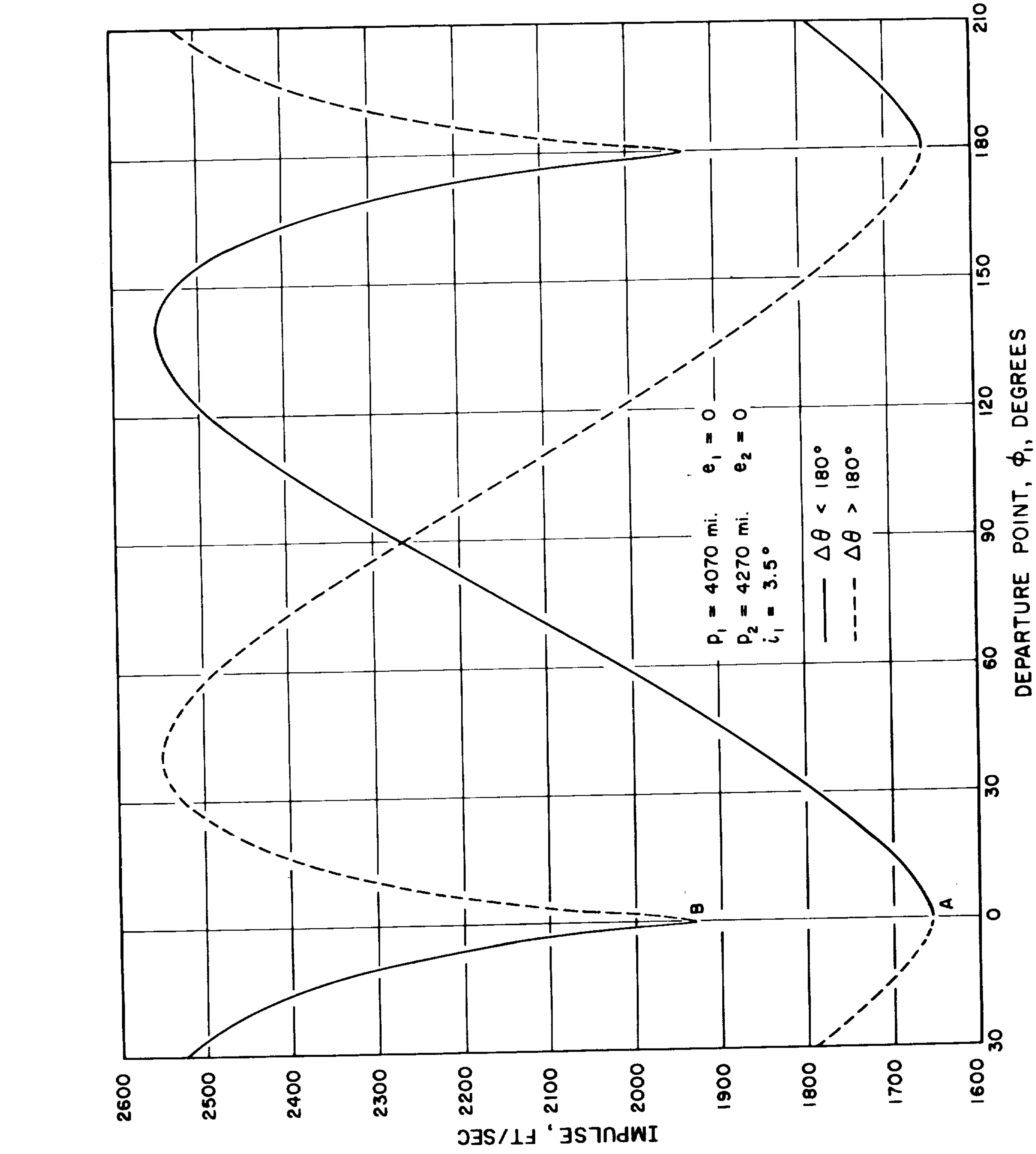


FIG.2. IMPULSE vs. DEPARTURE POINT

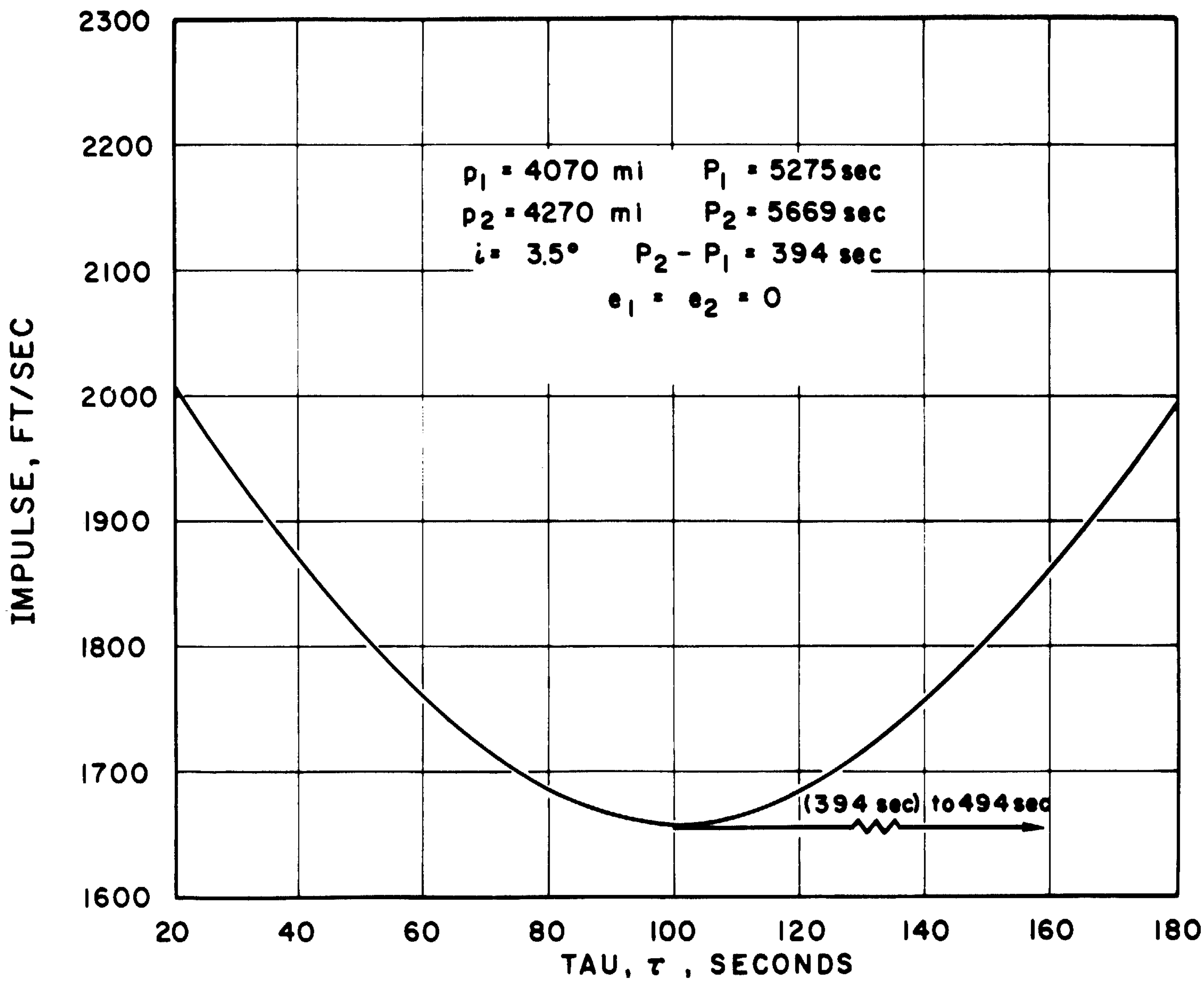


FIG. 3. IMPULSE vs. RELATIVE PHASE AT NODE FOR RENDEZVOUS

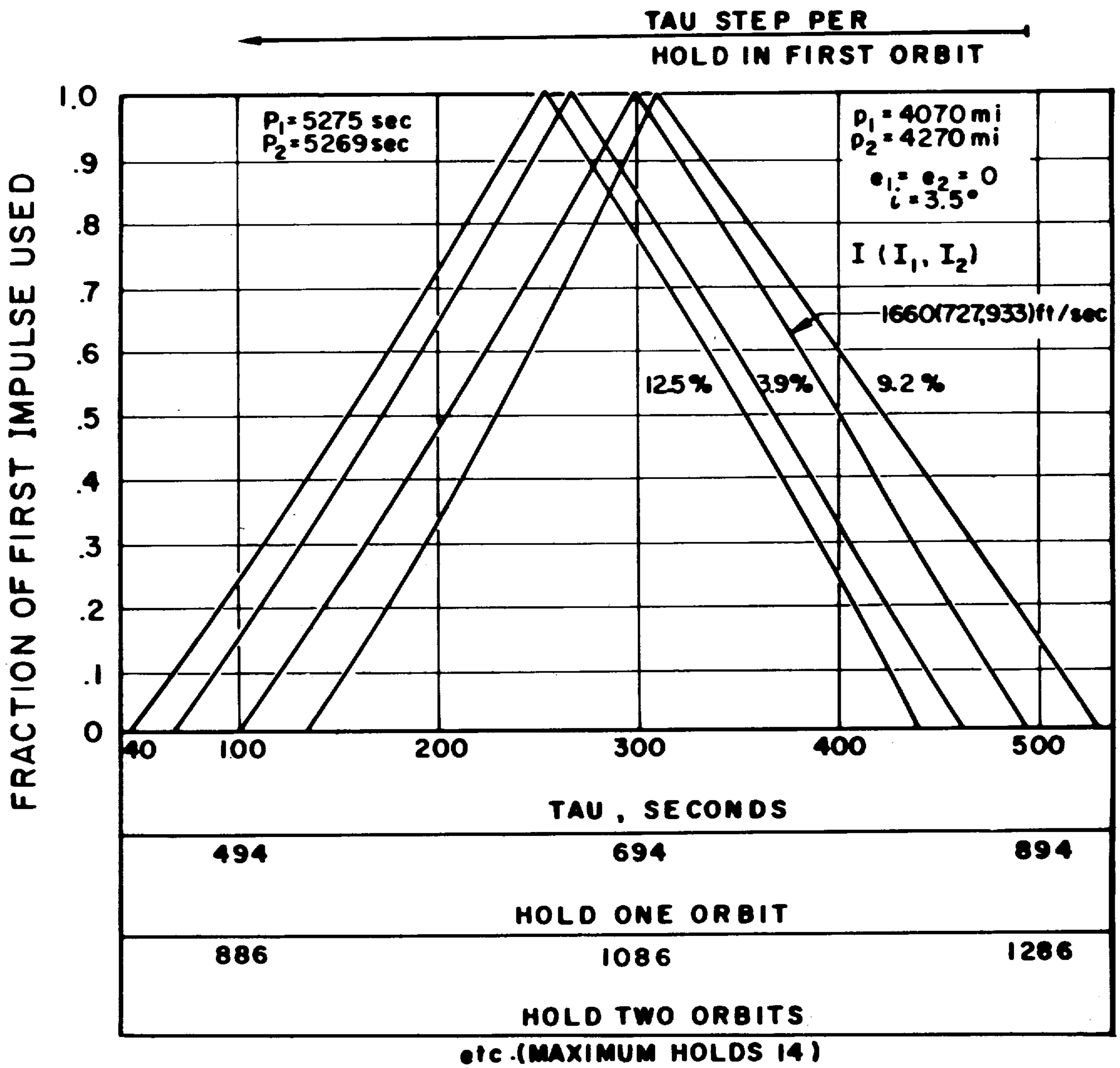


FIG. 4. THREE IMPULSE RENDEZVOUS.
(INCLINED CIRCULAR ORBIT CASE)

split, but if it slopes upward to the right the second is to be split. The whole range of τ which has to be covered is P_2 and as shown in Figure 5 this may require as many as fourteen orbit holds before initiating the rendezvous portion if the worst possible phase relation should develop as an initial condition. For the optimum case, the total impulse and the values of impulse at departure and arrival are indicated.

The second pair of orbits are two inclined ellipses with elements:

$p_1 = 5{,}000$ mi.	$e_1 = .2$	$_1 = -90°$	$i_1 = 5°$
$p_2 = 6{,}000$ mi.	$e_2 = .2$	$_2 = +30°$	$i_2 = 0°$

The possibilities for rendezvous are much better. In the first place there are four optimum two-impulse transfers as shown in Figure 5. The two orbits would intersect if in a plane and actually pass quite close to one another at the ascending node. Each of the optimum is characterized by whether it is short or long ($\Delta\theta < \pi$ or $> \pi$) and internal or external to the two ellipses. The latter distinction applies also to the periods, that is, internal has $P_3 < P_1$ and external, $P_3 > P_2$.

In Figure 6 the results of the time constrained search are indicated for each of the four optima as well as the range of relative phase (τ) accessible to rendezvous by splitting one of the impulses. For any point of Figure 6 a range of P' lying outside the interval P_2 to P_1 is possible in two ways, that is, either the first or the second impulse may be split. The nature of the curves is illustrated in Figure 7.

Finally, in Figure 8 the splitting fraction is plotted for each of the optima and for the six encircled points of Figure 6---except that only one of the impulse splits is indicated over the overlapping range. The percentage impulse required over the lowest is about 15%. Again the total impulse for each optimum and the impulses at departure and arrival are indicated, and which impulse is split can be seen by reference to Figure 7. For this case only a narrow range of τ (1,700 seconds to 2,300 seconds) is not accessible to three-impulse rendezvous within the 15% limitation. For phases in this small region, a hold of one revolution in the initial orbit is required.

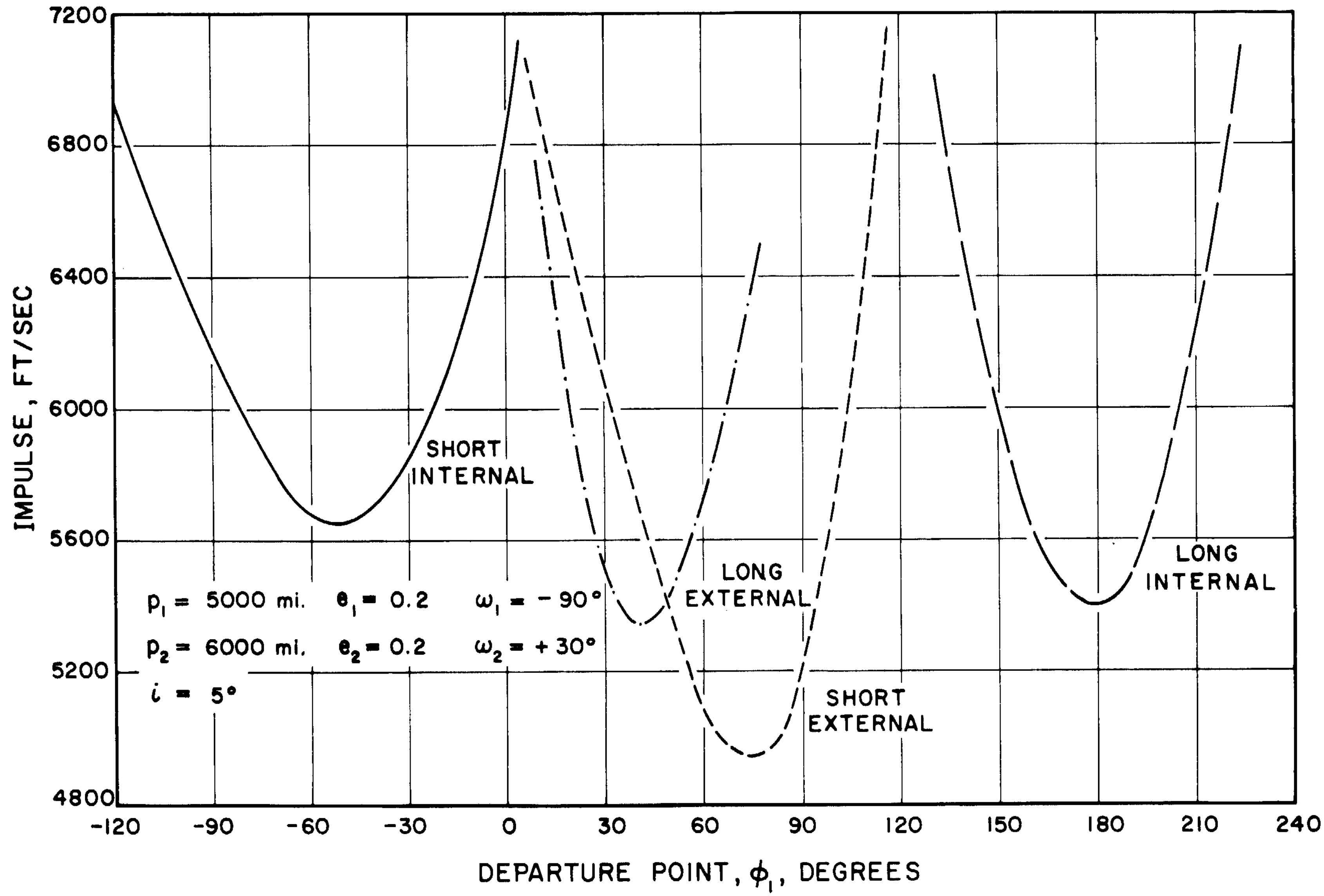


FIG. 5. IMPULSE vs. DEPARTURE POINT

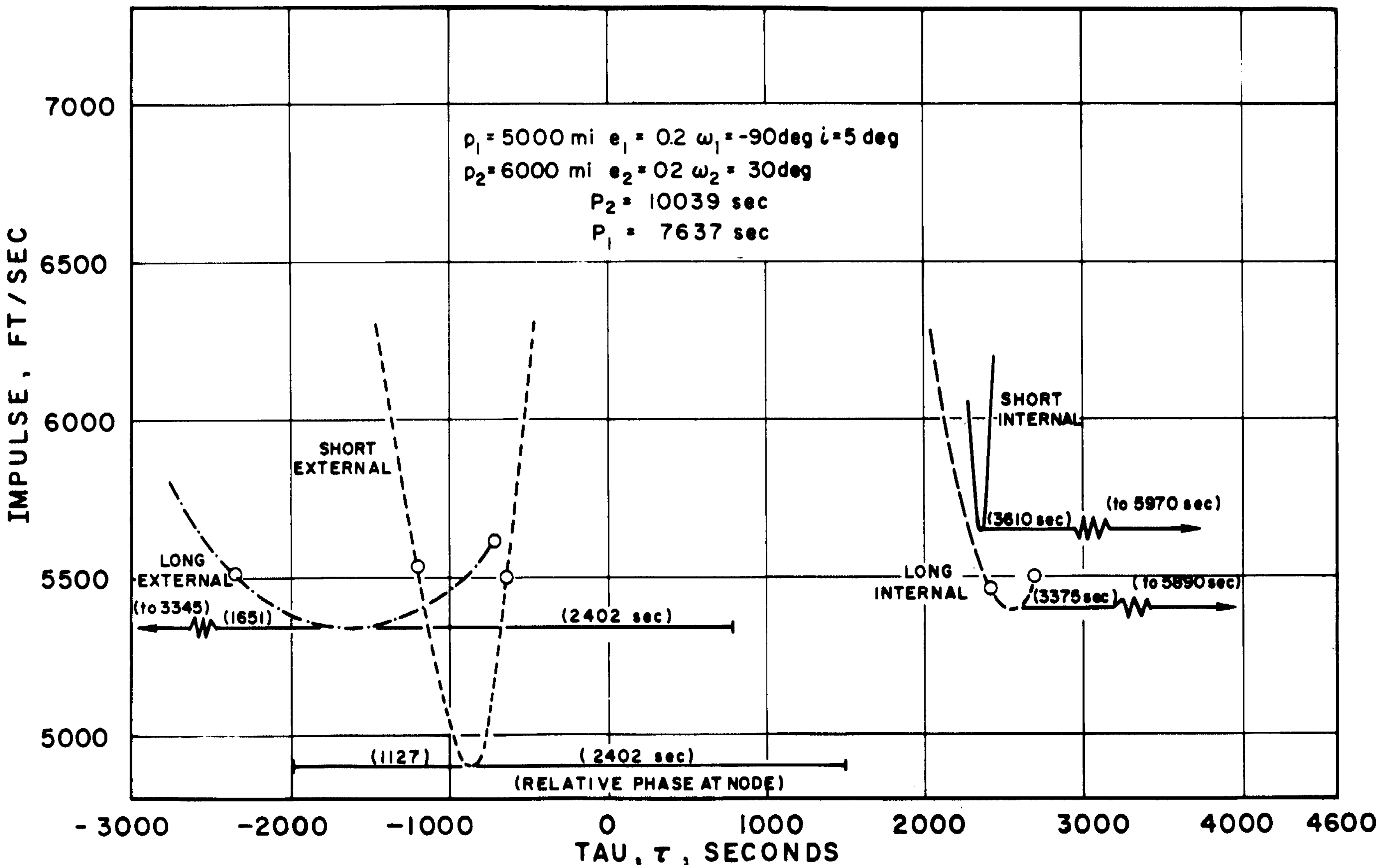


FIG. 6. IMPULSE VERSUS RELATIVE PHASE AT NODE FOR RENDEZVOUS

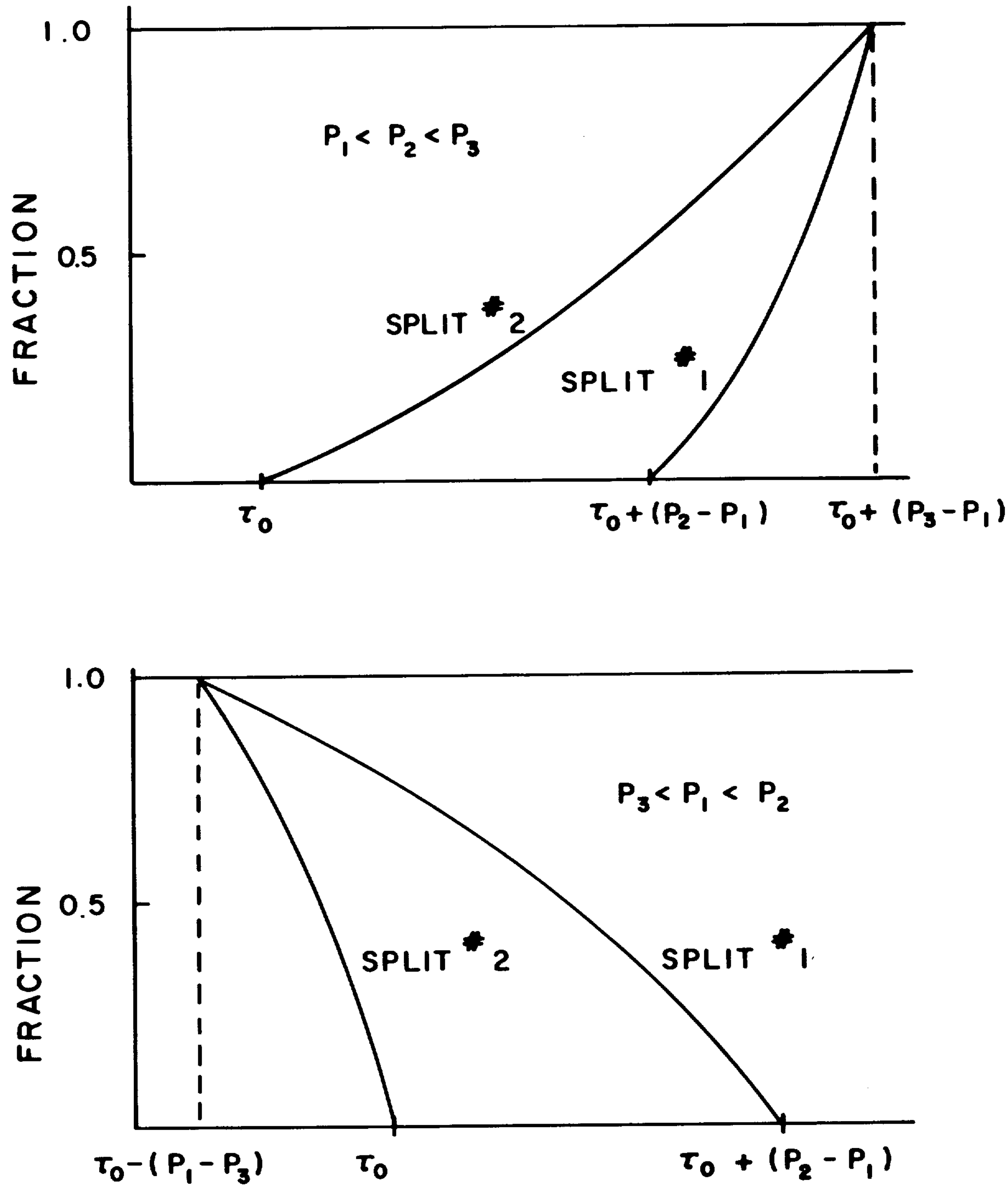


FIG. 7. NATURE OF IMPULSE SPLITTING CURVES WHEN P_3 LIES OUTSIDE P_1 TO P_2

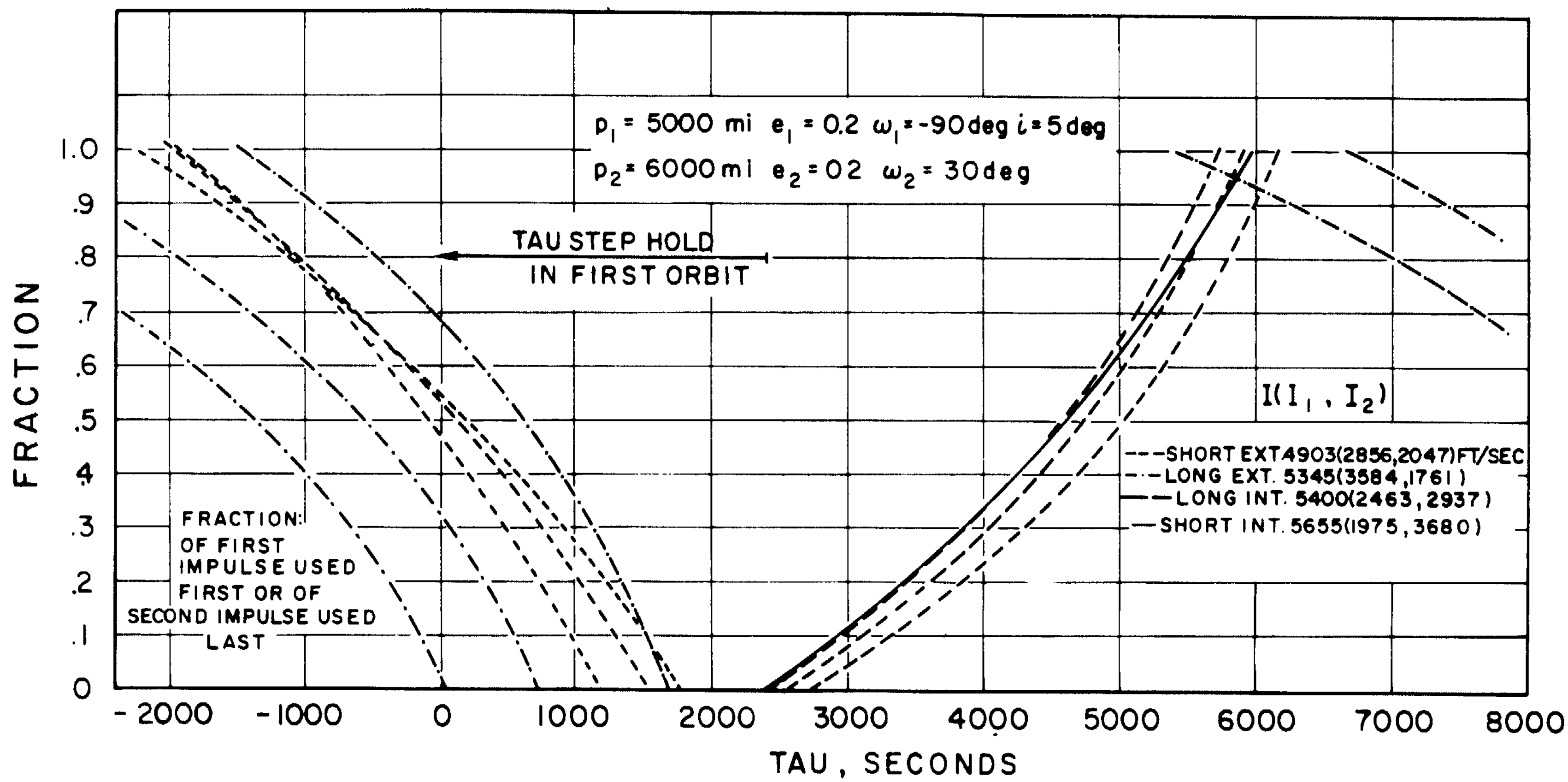


FIG. 8. THREE IMPULSE RENDEZVOUS

REFERENCES

1. (a) Kerfoot, H. P. and DesJardins, P. R., "Coplanar Two-Impulse Orbital Transfers," ARS Preprint 2063-61; or
(b) DesJardins, P. R., Bender, D. F. and Kerfoot, H. P., "Analytical Study of Satellite Rendezvous," (Final Report) North American Aviation Report No. MD 59-272, October 20, 1960.

2. McCue, G. A., "Optimum Two-Impulse Orbital Transfer and Rendezvous Between Elliptical Orbits," North American Aviation Report No. SID 62-1400; and portions of Ref. 3.

3. DesJardins, P. R., Bender, D. F. and McCue, G. A., "Orbital Transfer and Satellite Rendezvous," North American Aviation Report No. SID 62-870, August 31, 1962.

4. Hoelker, R. F. and Silber, R., "Analysis of Injection Schemes for Obtaining a Twenty-four Hour Orbit," Aerospace Engineering, Vol. 20, p. 29, January 1961.

REPUBLIC AVIATION CORPORATION

APPROXIMATION OF THE RESTRICTED PROBLEM BY THE TWO-FIXED-CENTER PROBLEM

By

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ACKNOWLEDGEMENT

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NOTATIONS

A	=	origin of rotating coordinate system
$\underline{A}$	=	position vector from barycenter to center of the rotating system
$\underline{A}_E$	=	the position vector of A relative to the earth
$\underline{E}$	=	position vector of the earth relative to the barycenter at t = 0
$\underline{E}'$	=	position vector of the earth relative to the barycenter, but rotated through an angle ωT
$\Delta\underline{E}$	=	$\underline{E}' - \underline{E}$
J	=	Hamiltonian (Jacobi integral) for the restricted problem
J*	=	difference between the restricted Hamiltonian and the two-fixed-center Hamiltonian
J_1	=	the part of $\bar{J}$ independent of α, β, and γ
J_2	=	the part of $\bar{J}$ that is a function of α, β, and γ
$\bar{J}$	=	Hamiltonian equivalent to J* but written in terms of two-fixed-center coordinates and momenta
J**	=	time dependent part of J*
J′	=	Hamiltonian of two-fixed-center problem
ℓ	=	length of position vector from earth to moon
$\underline{L}$	=	position vector from earth to moon
$\underline{\bar{L}}$	=	position vector of the moon relative to the earth in the rotating system
$\underline{\dot{L}}$	=	velocity of moon with respect to the earth ($\underline{\Omega} \times \underline{L}$)
$\underline{\dot{\bar{L}}}$	=	$\underline{\Omega} \times \underline{\bar{L}}$ in the rotating system
$\underline{P}_A$	=	momentum canonically conjugate to $\underline{\bar{R}}_A$
$\underline{P}'_A$	=	momentum canonically conjugate to $\underline{\bar{R}}'_A$
$\underline{R}$	=	position vector relative to a point fixed in inertial space e.g. barycenter
$\underline{R}_1$	=	position vector relative to the earth
$\underline{R}_2$	=	position vector relative to the moon
$\underline{\bar{R}}_A$	=	position vector relative to A in the rotating system
$\underline{\bar{R}}_1$	=	position vector relative to the earth in the rotating system
$\underline{\bar{R}}_2$	=	position vector relative to the moon in the rotating system

NOTATIONS (Cont'd)

$\underline{R}_E$	=	position vector from barycenter to earth
$\underline{R}_m$	=	position vector from barycenter to moon
$\underline{R}'_A$	=	position vector relative to A in the rotating system for the two-fixed-center problem
$\underline{R}_A$	=	position vector relative to point at A
r_1	=	length of position vector relative to earth
r_2	=	length of position vector relative to moon
T	=	a specific period of time
t	=	time variable
α	=	constant coefficient of $\underline{L}$ in composition of $\underline{A}$
β	=	constant coefficient of $\underline{\Omega}$ in composition of $\underline{A}$
γ	=	constant coefficient of $\underline{L}$ in composition of $\underline{A}$
θ	=	the angle of rotation of the coordinate system about the barycenter after a time T
μ	=	gravitational constant of the earth
μ'	=	gravitational constant of the moon
$\underline{\Omega}$	=	angular velocity vector of the moon about the earth
ω	=	the magnitude of angular velocity $\underline{\Omega}$
$\mathrm{grad}_{\underline{V}}$	=	gradient with respect to the components of $\underline{V}$ taken as coordinates

<u>Subscripts</u>

B	=	vector relative to the barycenter
0	=	initial value

<u>Superscript</u>

dot over quantity	=	first total time derivative
2 dots over quantity	=	second total time derivative

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Approximation of the Restricted Problem by the Two-Fixed Center Problem

By Mary Payne

16808

SUMMARY

In this report, a perturbation theory of the two-fixed-center problem leading to an approximation for the restricted-three-body problem is developed. It makes use of a generalization of the method developed at MSFC by Schulz-Arenstorff, Davidson, and Sperling. (1) The derivations are carried out in a coordinate system rotating about an accelerated origin, and the generalization consists of the selection of this origin in such a way as to minimize the effects of the non-integrable terms in the perturbation equations. The results of some numerical calculations are presented.

INTRODUCTION

The equations of motion for a vehicle moving in the gravitational fields of the earth and moon are:

$$\ddot{\underline{R}} = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \tag{1}$$

where $\underline{R}_1$, $\underline{R}_2$, and $\underline{R}$ are the position vectors of the vehicle referred to the earth, the moon, and a point fixed in inertial space, respectively. Lower case letters denote the magnitude of the corresponding vectors. In this report it will be assumed that the earth and moon are moving in circles, under their mutual gravitational attraction, about their common center of mass. This problem is the restricted three-body problem, and the fixed point may be taken to be the center of mass of the earth and the moon. An approximation to the solution of the restricted problem will be sought in terms of the known solution[3] to the Euler problem of two fixed centers of gravitation. The method will, in many respects, follow closely that developed by Schulz-Arenstorff, Davidson, and Sperling.[1] In their procedure, the problem is transformed to a coordinate system rotating about the center of mass. In this rotating system, the Euler problem is taken as the basis of a perturbation theory. Using the initial conditions of the Euler problem as a set of canonical variables, it is shown that[2]

$$\dot{\underline{R}}_0 = +\,\mathrm{grad}_{\underline{P}_0} J^*$$

and

$$\dot{\underline{P}}_0 = -\,\mathrm{grad}_{\underline{R}_0} J^* , \tag{2}$$

where $\underline{R}_0$ is the initial position vector in the rotating system, $\underline{P}_0$ is the momentum vector conjugate to $\underline{R}_0$, and J^* is the difference between the Hamiltonian for the restricted problem (Jacobi integral) and that for the Euler problem, and is given by

$$J^* = \underline{\Omega} \cdot \underline{R}_0 \times \underline{P}_0 + J^{**} . \tag{3}$$

The solution of the restricted problem is given in terms of an osculating two-fixed center problem with varying initial conditions. If J^{**} were zero, the equations for $\dot{\underline{R}}_0$ and $\dot{\underline{P}}_0$ could be integrated directly. In the Schulz-Arenstorff theory, J^{**} does not vanish and, in fact, contributes appreciably to the variation of $\underline{R}_0$ and $\underline{P}_0$ if the time interval over which the integration extends is too large, or if either the earth or the moon are approached closely by the vehicle during this time interval.

It is the purpose of this report to show that the effect of J^{**} can be reduced by selecting an origin for the rotating system other than the center of mass of the earth and moon. In the course of this development the details of the Schulz-Arenstorff method will be given, and the coordinates for a center of rotation will be determined so that J^{**} and its first time derivative vanish initially.

PRELIMINARY CONSIDERATIONS

Since the two-fixed center problem will be used as the basis of a perturbation theory, it is necessary that the earth and the moon be fixed in the rotating coordinate system. This implies that the origin of this rotating system must be fixed relative to the earth and the moon. The most general of such points will rotate about the barycenter with the angular velocity of the earth and the moon. The radius vector from the barycenter to the origin of the rotating system can be expressed as

$$\underline{A} = \alpha \underline{L} + \beta \underline{\Omega} + \gamma \dot{\underline{L}} , \tag{4}$$

where $\underline{L}$ and $\dot{\underline{L}}$ are the position and velocity vectors, respectively, of the moon relative to the earth in a non-rotating coordinate system, and $\underline{\Omega}$ is the angular velocity of the moon about the earth. From the definition of $\underline{L}$ and $\dot{\underline{L}}$ it is apparent that both vectors are known functions of time. Furthermore, $\underline{L}$ and $\dot{\underline{L}}$ are constant vectors in the rotating system and $\underline{\Omega}$ is constant in both the inertial frame and the rotating system. Thus, the requirement that the point A be fixed relative to the earth and the moon implies that α, β, and γ are numerical constants. The constant β may be chosen arbitrarily, for the point A is used to determine an axis of rotation oriented in the $\underline{\Omega}$ direction, and all points with the same α and γ will lie on the same axis independently of β. Thus, β may be taken as zero without loss of generality, and it will no longer appear in the formulation. Referring to Figure 1, it is seen that $\underline{R}$, $\underline{R}_1$, $\underline{R}_2$, $\underline{L}$, and $\underline{R}_A$, the position vector of the vehicle relative to A, satisfy the following relations:

$$\underline{R}_E = -\frac{\mu'}{\mu + \mu'} \underline{L} \tag{5}$$

$$\underline{R}_M = \frac{\mu}{\mu + \mu'} \underline{L} \tag{6}$$

$$\underline{R}_1 - \underline{R}_2 = \underline{L} \tag{7}$$

$$\underline{R}_A = \underline{R}_1 + \underline{R}_E - \underline{A} = \underline{R}_1 - \left(\alpha + \frac{\mu'}{\mu + \mu'}\right) \underline{L} - \gamma \dot{\underline{L}} \tag{8}$$

$$\underline{R}_A = \underline{R}_2 + \underline{R}_M - \underline{A} = \underline{R}_2 - \left(\alpha - \frac{\mu}{\mu + \mu'}\right) \underline{L} - \gamma \dot{\underline{L}} \tag{9}$$

$$\underline{R} = \underline{A} + \underline{R}_A = \underline{R}_A + \alpha \underline{L} + \gamma \dot{\underline{L}} \tag{10}$$

First, it is necessary to eliminate $\underline{R}$ from Eq. (1) and obtain the equations of motion in terms of $\underline{R}_A$, $\underline{R}_1$, and $\underline{R}_2$. To do this, one may differentiate Eq. (10) twice with respect to time:

$$\ddot{\underline{R}} = \ddot{\underline{R}}_A + \alpha\,\ddot{\underline{L}} + \gamma\,\dddot{\underline{L}}\ . \tag{11}$$

Now, the condition that the earth and moon move in circles under their mutual gravitational attraction means that

$$\dot{\underline{L}} = \underline{\Omega} \times \underline{L}$$

and

$$\ddot{\underline{L}} = \underline{\Omega} \times \dot{\underline{L}} = -(\mu + \mu')\,\frac{\underline{L}}{\ell^3}\ . \tag{12}$$

Differentiation of Eq. (12) (with $\dot{\ell} = 0$, as $\underline{L}$ has constant magnitude), enables us to write Eq. (11) as

$$\ddot{\underline{R}} = \ddot{\underline{R}}_A - \frac{\mu + \mu'}{\ell^3}\left[\alpha\,\underline{L} + \gamma\,\dot{\underline{L}}\right], \tag{13}$$

and the equations of motion (1) become

$$\ddot{\underline{R}}_A = -\mu\,\frac{\underline{R}_1}{r_1^3} - \mu'\,\frac{\underline{R}_2}{r_2^3} + \frac{\mu + \mu'}{\ell^3}\,(\alpha\,\underline{L} + \gamma\,\dot{\underline{L}})\ . \tag{14}$$

It should be noted that, at this stage, the coordinate system associated with A is an accelerated system since the origin has uniform circular motion. It is, however, not a rotating system yet - that is, the coordinate axes remain parallel to the inertial axes at the barycenter.

The next step is to transform to rotating coordinates about A. The vectors in this system will be denoted by bars, and the equations of motion become

$$\ddot{\bar{\underline{R}}}_A = -\mu\,\frac{\bar{\underline{R}}_1}{r_1^3} - \mu'\,\frac{\bar{\underline{R}}_2}{r_2^3} + \frac{\mu + \mu'}{\ell^3}\,(\alpha\,\bar{\underline{L}} + \gamma\,\dot{\bar{\underline{L}}}) - 2\,\underline{\Omega} \times \dot{\bar{\underline{R}}}_A - \underline{\Omega} \times \left(\underline{\Omega} \times \bar{\underline{R}}_A\right). \tag{15}$$

It should be noted that, in this rotating coordinate system, the earth and the moon are fixed, with position vector $\bar{\underline{L}}$ of the moon relative to the earth as a constant vector. The vector $\dot{\bar{\underline{L}}}$ does not represent the velocity of the moon (which is zero), but is a vector mutually perpendicular to $\underline{\bar{L}}$ and $\underline{\Omega}$, and satisfying Eq. (16) with bars over the vectors. As the rotating system has angular velocity $\underline{\Omega}$, it follows of course, that $\underline{\bar{\Omega}}$ and $\underline{\Omega}$ are identical.

A constant of motion for the problem in the rotating system may now be obtained by dotting Eq. (15) with $\dot{\underline{\bar{R}}}_A$ and noting that the earth and the moon are fixed in this system, so that

$$\dot{\underline{\bar{R}}}_A = \dot{\underline{\bar{R}}}_1 = \dot{\underline{\bar{R}}}_2 \,. \tag{16}$$

Thus,

$$\dot{\underline{\bar{R}}}_A \cdot \ddot{\underline{\bar{R}}}_A = \frac{d}{dt}\left(\frac{\dot{\underline{\bar{R}}}_A^{\,2}}{2}\right) = -\mu_1 \frac{\dot{\underline{\bar{R}}}_1 \cdot \dot{\underline{\bar{R}}}_1}{r_1^{\,3}} - \mu' \frac{\underline{\bar{R}}_2 \cdot \dot{\underline{\bar{R}}}_2}{r_2^{\,3}} \tag{17}$$

$$+ \frac{\mu + \mu'}{\ell^3}\left(\alpha\, \dot{\underline{\bar{R}}}_A \cdot \underline{\bar{L}} + \gamma\, \dot{\underline{\bar{R}}}_A \cdot \bar{\dot{\underline{L}}}\right) + \left(\underline{\Omega} \times \dot{\underline{\bar{R}}}_A\right) \cdot \left(\underline{\Omega} \times \underline{\bar{R}}_A\right)$$

$$= \frac{d}{dt}\left[\frac{\mu}{r_1} + \frac{\mu'}{r_2} + \frac{\mu + \mu'}{\ell^3}\left(\alpha\, \underline{\bar{R}}_A \cdot \underline{\bar{L}} + \gamma \underline{\bar{R}}_A \cdot \bar{\dot{\underline{L}}}\right) + \tfrac{1}{2}\left(\underline{\Omega} \times \underline{\bar{R}}_A\right)^2\right.$$

as $\underline{\bar{L}}$ and $\bar{\dot{\underline{L}}}$ are constant vectors. Denoting the constant of motion by J:

$$J = \tfrac{1}{2}\, \dot{\underline{\bar{R}}}_A^{\,2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \frac{\mu + \mu'}{\ell^3}\left(\alpha \underline{\bar{R}}_A \cdot \underline{\bar{L}} + \gamma\, \underline{\bar{R}}_A \cdot \bar{\dot{\underline{L}}}\right) - \tfrac{1}{2}\left(\underline{\Omega} \times \underline{\bar{R}}_A\right)^2. \tag{18}$$

It may now be shown that, if the vector

$$\underline{P}_A = \dot{\underline{\bar{R}}}_A + \underline{\Omega} \times \underline{\bar{R}}_A \tag{19}$$

is regarded as the momentum conjugate to $\underline{\bar{R}}_A$, the integral J of the motion becomes the Hamiltonian. To prove this, substitute for $\dot{\underline{\bar{R}}}_A$, using Eq. (19), in Eq. (18):

$$J = \tfrac{1}{2}\left(\underline{P}_A - \underline{\Omega} \times \underline{\bar{R}}_A\right)^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \frac{\mu + \mu'}{\ell^3}\left(\alpha \underline{\bar{R}}_A \cdot \underline{\bar{L}} + \gamma \underline{\bar{R}}_A \cdot \bar{\dot{\underline{L}}}\right) - \tfrac{1}{2}\left(\underline{\Omega} \times \underline{\bar{R}}_A\right)^2 \tag{20}$$

$$= \tfrac{1}{2}\, \underline{P}_A^{\,2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \underline{\Omega} \cdot \underline{\bar{R}}_A \times \underline{P}_A - \frac{\mu + \mu'}{\ell^3}\left(\alpha\, \underline{\bar{R}}_A \cdot \underline{\bar{L}} + \gamma\, \underline{\bar{R}}_A \cdot \bar{\dot{\underline{L}}}\right).$$

If $\underline{\bar{R}}_A$ and $\underline{P}_A$ are conjugate vectors, Hamilton's equations,

$$\dot{\underline{\bar{R}}}_A = \text{grad}_{\underline{P}_A} J = \underline{P}_A - \underline{\Omega} \times \underline{\bar{R}}_A , \tag{21}$$

and

$$\dot{\underline{P}}_A = -\text{grad}_{\underline{\bar{R}}_A} J = \text{grad}_{\underline{\bar{R}}_A}\left(\frac{\mu}{r_1} + \frac{\mu'}{r_2}\right) - \underline{\Omega} \times \underline{P}_A + \frac{\mu + \mu'}{\ell^3}\left(\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}\right) \tag{22}$$

must be satisfied. It is evident that Eq. (21) is identical with Eq. (19), defining the relation between velocity $\dot{\underline{\bar{R}}}_A$ and the momentum $\underline{P}_A$ conjugate to $\underline{\bar{R}}_A$. Now, it will be shown that Eq. (22) reduces to the equations of motion (15) in the rotating system. First,

$$\text{grad}_{\underline{\bar{R}}_A} \frac{\mu}{r_1} = -\frac{\mu}{r_1^{\,2}} \text{grad}_{\underline{\bar{R}}_A} r_1 . \tag{23}$$

But,

$$r_1^{\,2} = \underline{\bar{R}}_1 \cdot \underline{\bar{R}}_1 ; \tag{24}$$

hence,

$$\begin{aligned} 2 r_1 \text{grad}_{\underline{\bar{R}}_A} r_1 &= \text{grad}_{\underline{\bar{R}}_A} r_1^{\,2} \\ &= \text{grad}_{\underline{\bar{R}}_A} \left(\underline{\bar{R}}_A - \underline{\bar{R}}_E + \underline{\bar{A}}\right)^2 \\ &= \text{grad}_{\underline{\bar{R}}_A} \left[\underline{\bar{R}}_A^{\,2} + 2 \underline{\bar{R}}_A \cdot \left(\underline{\bar{A}} - \underline{\bar{R}}_E\right) + \left(\underline{\bar{A}} - \underline{\bar{R}}_E\right)^2 \right] \\ &= 2\underline{\bar{R}}_A + 2\left(\underline{\bar{A}} - \underline{\bar{R}}_E\right) = 2 \underline{\bar{R}}_1 , \end{aligned} \tag{25}$$

so that, finally,

$$\text{grad}_{\underline{\bar{R}}_A} r_1 = \frac{\underline{\bar{R}}_1}{r_1}$$

and

$$\text{grad}_{\underline{\bar{R}}_A} \frac{\mu}{r_1} = -\frac{\mu \underline{\bar{R}}_1}{r_1^{\,3}} . \tag{26}$$

Similarly,

$$\text{grad}_{\overline{\underline{R}}_A} \frac{\mu}{r_2} = - \frac{\mu \overline{\underline{R}}_2}{r_2^{\ 3}} , \tag{27}$$

so that Eq. (22) may be written as

$$\dot{\underline{P}}_A = - \frac{\mu \overline{\underline{R}}_1}{r_1^{\ 3}} - \frac{\mu' \overline{\underline{R}}_2}{r_2^{\ 3}} - \underline{\Omega} \times \underline{P}_A + \frac{\mu + \mu'}{\ell^3} \left(\alpha \, \overline{\underline{L}} + \gamma \, \dot{\overline{\underline{L}}} \right) . \tag{28}$$

Now, from Eq. (19),

$$\dot{\underline{P}}_A = \ddot{\overline{\underline{R}}}_A + \underline{\Omega} \times \dot{\overline{\underline{R}}}_A , \tag{29}$$

and use of this relation for $\dot{\underline{P}}_A$ and Eq. (19) for $\underline{P}_A$ in Eq. (28) yields

$$\ddot{\overline{\underline{R}}}_A + \underline{\Omega} \times \dot{\overline{\underline{R}}}_A = - \frac{\mu \overline{\underline{R}}_1}{r_1^{\ 3}} - \frac{\mu' \overline{\underline{R}}_2}{r_2^{\ 3}} - \underline{\Omega} \times \dot{\overline{\underline{R}}}_A - \underline{\Omega} \times \left(\underline{\Omega} \times \overline{\underline{R}}_A \right) + \frac{\mu + \mu'}{\ell^3} \left(\alpha \, \overline{\underline{L}} + \gamma \, \dot{\overline{\underline{L}}} \right) . \tag{30}$$

Finally, if the $\underline{\Omega} \times \dot{\overline{\underline{R}}}_A$ on the left is transposed to the right hand side of Eq. (30), it becomes identical with the equations of motion (15) in the rotating system.

RELATION BETWEEN THE TWO-FIXED CENTER PROBLEM AND THE RESTRICTED PROBLEM

A Hamiltonian, J, has now been obtained for the restricted problem in a rotating coordinate system with the origin at A:

$$J = \tfrac{1}{2} \underline{P}_A^{\ 2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \underline{\Omega} \cdot \overline{\underline{R}}_A \times \underline{P}_A - \frac{\mu + \mu'}{\ell^3} \left(\alpha \, \overline{\underline{R}}_A \cdot \overline{\underline{L}} + \gamma \, \overline{\underline{R}}_A \cdot \dot{\overline{\underline{L}}} \right) , \tag{31}$$

with

$$\overline{\underline{A}} = \alpha \, \overline{\underline{L}} + \gamma \, \dot{\overline{\underline{L}}} , \tag{32}$$

referred to the barycenter of earth and moon, and

$$\underline{P}_A = \dot{\overline{\underline{R}}}_A + \underline{\Omega} \times \overline{\underline{R}}_A \ . \tag{33}$$

The development so far differs slightly from that of Schulz-Arenstorff, Davidson, and Sperling(1) in two respects: it has been carried out in three dimensions instead of two, and the center of the rotating coordinate system is at A instead of the barycenter. Following their development, a solution of Eq. (31) in terms of the solution of the two-fixed center problem is now sought. For the two-fixed center problem, the Hamiltonian is given by:

$$J' = \tfrac{1}{2}\, \underline{P}_A'^{\,2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \ , \tag{34}$$

and the Hamilton equations are

$$\dot{\overline{\underline{R}}}_A' = \mathrm{grad}_{\underline{P}_A'} \, J' = \underline{P}_A'$$

and

$$\dot{\underline{P}}_A' = - \mathrm{grad}_{\overline{\underline{R}}_A} \, J' = - \mu \frac{\overline{\underline{R}}_1'}{r_1^3} - \mu' \frac{\overline{\underline{R}}_2'}{r_2^3} \ . \tag{35}$$

Denoting the solution of the two-fixed center problem by primes and that for the restricted problem without primes, the solution sought is to have the form

$$\underline{R}\left(\underline{R}_0, \underline{P}_0, t\right) = \underline{R}'\left(\underline{R}_0(t), \underline{P}_0(t), t\right)$$

and (36)

$$\underline{P}\left(\underline{R}_0, \underline{P}_0, t\right) = \underline{P}'\left(\underline{R}_0(t), \underline{P}_0(t), t\right) \ .$$

Thus, the problem is reduced to finding the time dependence of the initial conditions in the solution of the two-fixed center problem that provide the solution of the restricted problem in the same functional form as that of the two-fixed center solution.

The theorem, mentioned in the introduction, on the equations determining the time variation of the initial conditions will now be given a precise statement.

Theorem: If $\underline{R}(\underline{R}_0, \underline{P}_0, t)$ and $\underline{P}(\underline{R}_0, \underline{P}_0, t)$ constitute the solution of a problem with Hamiltonian $J(\underline{R}, \underline{P})$ while $\underline{R}'(\underline{R}_0, \underline{P}_0, t)$ and $\underline{P}'(\underline{R}_0, \underline{P}_0, t)$ constitute the solution of a problem with Hamiltonian $J'(R', P')$ with

$$\underline{R}(\underline{R}_0, \underline{P}_0, 0) = \underline{R}'(\underline{R}_0, \underline{P}_0, 0) = \underline{R}_0$$

and

$$\underline{P}(\underline{R}_0, \underline{P}_0, t) = \underline{P}'(\underline{R}_0, \underline{P}_0, 0) = \underline{P}_0 \tag{37}$$

then Eqs. (36) are satisfied with $\underline{R}_0(t)$ and $\underline{P}_0(t)$, determined by the equations

$$\dot{\underline{R}}_0(t) = \text{grad}_{\underline{P}_0} J^*(\underline{R}_0, \underline{P}_0, t)$$

and

$$\dot{\underline{P}}_0(t) = -\text{grad}_{\underline{R}_0} J^*(\underline{R}_0, \underline{P}_0, t), \tag{38}$$

where

$$\bar{J}(\underline{R}', \underline{P}') = J(\underline{R}', \underline{P}') - J'(\underline{R}', \underline{P}') = J^*(\underline{R}_0, \underline{P}_0, t) \tag{39}$$

Wherever $\underline{R}_0$ and $\underline{P}_0$ occur on the right hand side as a result of the gradient operations, they are to be replaced by $\underline{R}_0(t)$ and $\underline{P}_0(t)$, respectively.

This theorem has been proven by Arenstorf[2] in an unpublished note and will now be applied.

To obtain the differential equations for $\underline{R}_0(t)$ and $\underline{P}_0(t)$, $\bar{J}$ must be written in terms of $\bar{\underline{R}}'_A$ and $\underline{P}'_A$, associated with the two-fixed outer problem. That is,

$$\begin{aligned} \bar{J} &= J(\bar{\underline{R}}'_A, \underline{P}'_A) - J'(\bar{\underline{R}}'_A, \underline{P}'_A) \\ &= -\underline{\Omega} \cdot \bar{\underline{R}}'_A \times \underline{P}'_A - \frac{\mu + \mu'}{\ell^3} (\alpha \bar{\underline{R}}'_A \cdot \underline{L} + \gamma \bar{\underline{R}}'_A \cdot \underline{L}'), \end{aligned} \tag{40}$$

where $J(\bar{\underline{R}}'_A, \underline{P}'_A)$ is obtained from Eq. (31) by replacing $\bar{\underline{R}}_A$ and $\underline{P}_A$ by the corresponding primed quantities, and $J'(\bar{\underline{R}}'_A, \underline{P}'_A)$ is given by Eq. (34).

It is now necessary to obtain J^* by expressing $\bar{J}$ in terms of the initial conditions of the two-fixed center problem. This is very difficult to do exactly, as the solution[3] of the two-fixed center problem is given in terms of elliptic functions with the initial conditions entering not only in coefficients of these functions but also in their moduli. Therefore, the solution of the two-fixed center problem is a transcendental function of the initial conditions. An approximate solution is, however, obtainable by expanding $\bar{J}$ as a power series in time:

$$\begin{aligned} \bar{J} &= \bar{J}(0) + \dot{\bar{J}}(0)t + \ddot{\bar{J}}(0)\frac{t^2}{2} + \dots \\ &= J^*(0) + \dot{J}^*(0)t + \ddot{J}^*(0)\frac{t^2}{2} + \dots \quad . \end{aligned} \tag{41}$$

Using Eq. (40), the first time derivative of $\bar{J}$ is

$$\dot{\bar{J}} = -\underline{\Omega} \cdot \dot{\underline{\bar{R}}}'_A \times \underline{P}'_A - \underline{\Omega} \cdot \underline{\bar{R}}'_A \times \dot{\underline{P}}'_A - \frac{\mu+\mu'}{\ell^3}(\alpha \dot{\underline{\bar{R}}}_A \cdot \underline{L} + \gamma \dot{\underline{\bar{R}}}'_A \cdot \dot{\underline{\bar{L}}}). \tag{42}$$

Now, Eq. (42) contains time derivatives of $\underline{\bar{R}}_A$ and $\underline{P}_A$, which may be eliminated by means of the Hamilton equations (35) for the two-fixed center problem:

$$\dot{\bar{J}} = -\underline{\Omega} \cdot \underline{P}'_A \times \underline{P}'_A - \underline{\Omega} \cdot \underline{\bar{R}}' \times \left(-\frac{\mu \underline{\bar{R}}'_1}{r_1^3} - \frac{\mu' \underline{\bar{R}}_2}{r_2^3}\right) - \frac{\mu+\mu'}{\ell^3}\underline{P}'_A \cdot (\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}) . \tag{43}$$

The first term in this equation vanishes. Evaluation of $\bar{J}$ and $\dot{\bar{J}}$ at t=0 yields

$$\bar{J}(0) = J^*(0) = -\underline{\Omega} \cdot \underline{\bar{R}}'_{A0} \times \underline{P}'_{A0} - \frac{\mu+\mu'}{\ell^3} \underline{\bar{R}}'_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}). \tag{44}$$

and

$$\dot{\bar{J}}(0) = \dot{J}^*(0) = -\underline{\Omega} \cdot \underline{\bar{R}}'_{A0} \times \left(\frac{-\mu \underline{\bar{R}}'_{10}}{r_{10}^3} - \frac{\mu' \underline{\bar{R}}'_{20}}{r_{20}^3}\right) - \frac{\mu+\mu'}{\ell^3} \underline{P}'_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}) \tag{45}$$

Setting

$$J_1 = -\underline{\Omega} \cdot \underline{\bar{R}}'_{A0} \times \underline{P}'_{A0} \tag{46}$$

and

$$\begin{aligned} J_2 = &-\frac{\mu+\mu'}{\ell^3} \underline{\bar{R}}'_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}) + t\Big[\underline{\Omega} \cdot \underline{\bar{R}}'_{A0} \times \left(\frac{\mu \underline{\bar{R}}'_{10}}{r_{10}^3} + \frac{\mu' \underline{\bar{R}}'_{20}}{r_{20}^3}\right) \\ &- \frac{\mu+\mu'}{\ell^3} \underline{P}'_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}) \Big] + \dots , \end{aligned} \tag{47}$$

so that

$$J^* = J_1 + J_2 .$$

Application of the Arenstorf theorem, now yields

$$\dot{\bar{\underline{R}}}'_{A0} = + \operatorname{grad}_{\underline{P}'_{A0}} J^* = - \underline{\Omega} \times \bar{\underline{R}}'_{A0} + \operatorname{grad}_{\underline{P}'_{A0}} J_2 \tag{48}$$

and

$$\dot{\underline{P}}'_{A0} = - \operatorname{grad}_{\bar{\underline{R}}'_{A0}} J^* = - \underline{\Omega} \times \underline{P}'_{A0} - \operatorname{grad}_{\bar{\underline{R}}'_{A0}} J_2 \tag{49}$$

as the differential equations for the variation of the two-fixed center initial conditions, which must be included in the two-fixed center solution in order that it may become the solution of the restricted problem.

If J_2 were zero, Eqs. (48) and (49) would integrate immediately. They would simply say that $\underline{R}_{A0}$ and $\underline{P}_{A0}$ rotate clockwise with angular velocity $\underline{\Omega}$. That is, in the rotating system the solution of the restricted problem at time T would be given by the solution of the two-fixed center problem at time T, with initial conditions obtained from those of the restricted problem by a clockwise rotation through $\underline{\Omega}\, T$ about the point A. For T=0, the restricted and two-fixed center problems have the same initial conditions and, hence, have exactly the same solution.

Actually, of course, J_2 does not vanish, and it is here that the selection of the point A enters. Every term of J_2 involves either $\bar{\underline{R}}_{A0}$ or $\underline{P}_{A0}$, which depend on the selection of the point A, so that this point should be selected so as to minimize the contribution of J_2 to the variation of the initial conditions. This could be done in various ways. Inasmuch as the position of the point A depends on the two parameters α and γ, it is evident that only two conditions can be imposed on the selection of A. Several such conditions suggest themselves immediately:

(1) Determine α and γ so that in J_2 the constant term and the coefficient of t vanish for the initial values of $\bar{\underline{R}}'_{A0}$ and $\underline{P}'_{A0}$.

(2) Determine α and γ so that J_2 vanish for t=0, with initial values of $\underline{R}'_{A0}$ and $\underline{P}'_{A0}$, and also vanish at t=T, with the rotated values of $\bar{\underline{R}}'_{A0}$ and $\underline{P}'_{A0}$ determined by J_1 at time T.

(3) Determine α and γ so that the square of J_2 is minimized over the time interval 0 to T, using either the initial values of $\bar{\underline{R}}'_{A0}$ and $\underline{P}'_{A0}$ or their time dependent values determined by J_1 over the interval.

The first method has the disadvantage that the validity of the approximation would deteriorate with time, and there is no obvious way of estimating the duration of validity. The other two methods have the disadvantage that, if the time interval specified is too long, the approximation would not be valid, even initially, and again, a criterion for "too long" is missing. It was, therefore, decided to try the first method, which would give some insight into the duration of validity, and might very well produce results of practical value.

DETERMINATION OF α AND γ

In accordance with the conclusion of the last section, α and γ are to be determined by the equations

$$\underline{\bar{R}}_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}) = 0 \tag{50}$$

and

$$\underline{\Omega} \cdot \underline{\bar{R}}_{A0} \times \left(\frac{\mu \underline{\bar{R}}_{10}}{r_{10}^3} + \mu' \frac{\underline{\bar{R}}_{20}}{r_{20}^3} \right) - \frac{\mu + \mu'}{\ell^3} \underline{P}_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}}) = 0, \tag{51}$$

so that the first two terms in the power series expansion of J_2 in Eq. (47) vanish. The primes have been omitted in Eqs. (50) and (51) because the initial values of $\underline{\bar{R}}'_{A0}$ and $\underline{P}'_{A0}$, regarded as variable parameters for the restricted problem, are the initial values of the restricted problem by the Arenstorf theorem.[2] Now, $\underline{\bar{R}}_{A0}$ and $\underline{P}_{A0}$ depend on the selection of the point A, so that, for the determination of α and γ from Eqs. (50) and (51), they should be replaced by the position and momentum of the vehicle relative to some point independent of A. A particularly compact form is obtained for the equations of α and γ by replacing $\underline{P}_{A0}$ by $\underline{P}_{10}$ and $\underline{\bar{R}}_{A0}$ by $\underline{\bar{R}}_{10}$ or $\underline{\bar{R}}_{20}$, as follows. First, since from Eq. (19)

$$\underline{P}_{A0} = \dot{\underline{\bar{R}}}_{A0} + \underline{\Omega} \times \underline{\bar{R}}_{A0} , \tag{52}$$

for any point A fixed relative to earth and moon, it follows that

$$\underline{P}_{10} = \dot{\underline{\bar{R}}}_{10} + \underline{\Omega} \times \underline{\bar{R}}_{10} . \tag{53}$$

Therefore, since in the rotating system the velocity of the vehicle relative to the earth is the same as that relative to A (both are fixed points in the rotating system),

$$\begin{aligned} \underline{P}_{A0} &= \underline{P}_{10} + \underline{\Omega} \times (\underline{\bar{R}}_{A0} - \underline{\bar{R}}_{10}) \\ &= \underline{P}_{10} - \underline{\Omega} \times \left(\left(\alpha + \frac{\mu'}{\mu + \mu'}\right) \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}} \right) \\ &= \underline{P}_{10} - \left(\alpha + \frac{\mu'}{\mu + \mu'} \dot{\underline{\bar{L}}} + \gamma \frac{\mu + \mu'}{\ell^3} \underline{\bar{L}} , \end{aligned} \tag{54}$$

on making use of Eqs. (8) and (12). Thus, the third term of Eq. (51) will be proportional to

$$\underline{P}_{A0} \cdot (\alpha \, \underline{\bar{L}} + \gamma \, \underline{\dot{\bar{L}}}) = \underline{P}_{10} \cdot (\alpha \, \underline{\bar{L}} + \gamma \, \underline{\dot{\bar{L}}}) - \gamma \frac{\mu'}{\ell} , \tag{55}$$

where the terms in $\alpha\gamma$ have cancled out.

Again using Eq. (8), the first term of Eq. (51) will involve

$$\begin{aligned} \underline{\Omega} \cdot \underline{\bar{R}}_{A0} \times \underline{\bar{R}}_{10} &= - \underline{\Omega} \times \left[(\alpha + \frac{\mu'}{\mu + \mu'}) \, \underline{\bar{L}} + \gamma \, \underline{\dot{\bar{L}}} \right] \cdot \underline{\bar{R}}_{10} \\ &= - \underline{\bar{R}}_{10} \cdot \left[(\alpha + \frac{\mu'}{\mu + \mu'}) \, \underline{\dot{\bar{L}}} - \gamma \frac{\mu + \mu'}{\ell^3} \, \underline{\bar{L}} \right] , \end{aligned} \tag{56}$$

and the second term will be proportional to

$$\begin{aligned} \underline{\Omega} \cdot \underline{\bar{R}}_{A0} \times \underline{\bar{R}}_{20} &= - \underline{\Omega} \times \left[(\alpha - \frac{\mu}{\mu + \mu'}) \, \underline{\bar{L}} + \gamma \, \underline{\dot{\bar{L}}} \right] \cdot \underline{\bar{R}}_{20} \\ &= - \underline{\bar{R}}_{20} \cdot \left[(\alpha - \frac{\mu}{\mu + u'}) \, \underline{\dot{\bar{L}}} - \gamma \frac{\mu + u'}{\ell^3} \, \underline{\bar{L}} \right] , \end{aligned} \tag{57}$$

so that Eq. (51) may now be written as follows:

$$\begin{aligned} &\frac{\mu}{r_{10}^{\;3}} \left[- (\alpha + \frac{\mu'}{\mu + \mu'}) \, \underline{\bar{R}}_{10} \cdot \underline{\dot{\bar{L}}} + \gamma \frac{\mu + u'}{\ell^3} \, \underline{\bar{R}}_{10} \cdot \underline{\bar{L}} \right] \\ &\quad + \frac{\mu'}{r_{20}^{\;3}} \left[- (\alpha - \frac{\mu}{\mu + \mu'}) \, \underline{\bar{R}}_{20} \cdot \underline{\dot{\bar{L}}} + \gamma \frac{\mu + \mu'}{\ell^3} \, \underline{\bar{R}}_{20} \cdot \underline{\bar{L}} \right] \\ &\quad - \frac{\mu + \mu'}{\ell^3} \left[\underline{P}_{10} \cdot (\alpha \, \underline{\bar{L}} + \gamma \, \underline{\dot{\bar{L}}}) - \frac{\mu'}{\ell} \gamma \right] = 0 \end{aligned} \tag{58}$$

or, collecting terms in α and γ:

$$
\begin{aligned}
&-\alpha\left[\bar{\underline{R}}_{10}\cdot\dot{\bar{\underline{L}}}\left(\frac{\mu'}{r_{10}^{3}}+\frac{\mu'}{r_{20}^{3}}\right)+\frac{\mu+\mu'}{\ell^{3}}\,\underline{P}_{10}\cdot\bar{\underline{L}}\right]\\
&+\frac{\mu+\mu'}{\ell^{3}}\,\gamma\left[\mu\,\frac{\bar{\underline{R}}_{10}\cdot\bar{\underline{L}}}{r_{10}^{3}}+\mu'\,\frac{\bar{\underline{R}}_{20}\cdot\bar{\underline{L}}}{r_{20}^{3}}+\frac{\mu'}{\ell}-\underline{P}_{10}\cdot\dot{\bar{\underline{L}}}\right]\\
&-\frac{\mu\,\mu'}{\mu+\mu'}\,\bar{\underline{R}}_{10}\cdot\dot{\bar{\underline{L}}}\left(\frac{1}{r_{10}^{3}}-\frac{1}{r_{20}^{3}}\right)\\
&=0
\end{aligned}
\tag{58}
$$

where use has been made of the fact that

$$
\bar{\underline{R}}_{10}\cdot\dot{\bar{\underline{L}}}=\bar{\underline{R}}_{20}\cdot\dot{\bar{\underline{L}}}\quad . \tag{59}
$$

Using Eq. (8) once more, one obtains for Eq. (50):

$$
\begin{aligned}
&\bar{\underline{R}}_{10}\cdot(\alpha\,\underline{L}+\gamma\dot{\bar{\underline{L}}})-\left[\left(\alpha+\frac{\mu'}{\mu+\mu'}\right)\bar{\underline{L}}+\gamma\dot{\bar{\underline{L}}}\right]\cdot\left[\alpha\,\bar{\underline{L}}+\gamma\dot{\bar{\underline{L}}}\right]\\
&=\bar{\underline{R}}_{10}\cdot(\alpha\,\bar{\underline{L}}+\gamma\dot{\bar{\underline{L}}})-\alpha\left(\alpha+\frac{\mu'}{\mu+\mu'}\right)\ell^{2}-\gamma^{2}\cdot\ell^{2}\\
&=-\alpha^{2}\ell^{2}+\alpha\left(\bar{\underline{R}}_{10}\cdot\bar{\underline{L}}-\frac{\mu'}{\mu+\mu'}\,\ell^{2}\right)-\gamma^{2}\frac{\mu+\mu'}{\ell}+\gamma\left(\bar{\underline{R}}_{10}\cdot\dot{\underline{L}}\right)=0
\end{aligned}
\tag{60}
$$

If Eqs. (58) and (60) are solved for α and γ , a point A is determined so that the following procedure should give an approximation to the restricted problem valid for a time interval whose length depends on the size of J* and the rate of variation of $\bar{\underline{R}}'_{10}$ and $\underline{P}'_{10}$. The procedure is carried out in the rotating system as follows:

> Modify the initial conditions of the restricted problem by a clockwise rotation through ω T about the point A, and solve the two-fixed center problem with these modified initial conditions. Then, $\bar{\underline{R}}'_{A}(T)$ and $\underline{P}'_{A}(T)$, given by the two-fixed center problem, should match $\bar{\underline{R}}_{A}(T)$ given by the restricted problem with unmodified initial conditions.

APPLICATION OF THE METHOD

In order to carry out a numerical test of the method, use was made of the Republic interplanetary trajectory program. The input for this program requires that initial conditions be given in a coordinate system with its origin at the earth and axes with fixed directions in space. The z-axis points towards the pole star, the x-axis points to the first point of Aries, and the y-axis is selected so that the system is orthogonal and right-handed. The output includes coordinates and velocities of the vehicle in this same system. An option is available which fixes the moon at any desired point on its orbit and computes a two-fixed center problem for this fixed position of the moon and given initial conditions. A set of initial conditions is available which yields a lunar trajectory (referred to, henceforth, as the base case) with a moving moon, starting near the earth, closely circling the moon and returning to the earth. Thus, to test the application one could modify the coordinates and velocities at various points on this base case and compute a two-fixed center problem from the modified conditions to obtain a comparison, which should indicate the time intervals over which the approximation is useful for various portions of the trajectory.

The modification of the initial conditions derived in the preceding sections was carried out in a rotating system, and it is now necessary to transform this modification for use in the coordinate system of the interplanetary program. To see how this may be done, suppose for the moment that the point A is at the barycenter, i.e., α and γ are both zero, and that the fixed and rotating systems are coincident at $t = 0$. It is evident, in this case, that the two-fixed center orbit obtained from the initial conditions, modified by a clockwise rotation through an angle θ about the barycenter, is exactly the same relative to the earth and moon as if the initial conditions had been unmodified and the earth and moon had been rotated counterclockwise through θ about the barycenter. Now, the angle θ is ωT, where T is the time at which the comparison is to be made. Hence, if the earth, the moon, and the two-fixed center orbit, corresponding to the modified initial conditions, is rigidly rotated counterclockwise through ωT, the earth and moon will coincide with their positions at time T in the fixed system, and the point corresponding to time T on the two-fixed center orbit is the one to be compared with the restricted problem carried out in the fixed system. Moreover, this counterclockwise rotation just transforms the two-fixed center problem, with modified initial conditions and earth and moon in initial position, into that with unmodified initial conditions and earth and moon in their T positions. Therefore, for α and γ both zero, the comparison can be made, using the interplanetary program by fixing the moon in its T position and referring the unmodified initial conditions to the coordinate system centered at the earth at time T. This is indicated in Fig. 2, where the unprimed initial conditions are referred to the earth at $t = 0$, and the primed initial conditions refer to the earth at $t = T$. The initial conditions are fixed.

A comment on the relation between the momentum vector $\underline{P}_B$, conjugate to $\bar{\underline{R}}_B$, and the velocity vector $\dot{\bar{\underline{R}}}_B$, where B is used to indicate that the barycenter is the origin of the rotating system, in now in order. Recalling the definition of P_A in Eq. (19), it follows that

$$\underline{P}_B = \dot{\bar{\underline{R}}}_B + \underline{\Omega} \times \bar{\underline{R}}_B \,, \tag{61}$$

and hence P_B is simply the velocity vector in the fixed system with its components referred to the instantaneous rotating axes. Since it has been assumed that the fixed and rotating systems are coincident at t = 0, it follows that

$$\underline{P}_{B_o} = \dot{\underline{R}}_{B_o} \,, \tag{62}$$

where $\dot{R}_B$ is in the fixed system (recall that bars denote rotating system). At time T, if the $\underline{P}_B$ vector is rotated through ω T counterclockwise, it will become the $\dot{\underline{R}}_B$ vector. But this is just the transformation that has been used to translate the two-fixed center approximation from the rotating to the fixed system.

Thus, if the barycenter is the origin of the rotating system (i.e., $\alpha = \gamma = 0$), the prescription for the approximation is the following:

(1) Let

$$\underline{\Delta E} = \underline{E}' - \underline{E} = \frac{-\mu'}{\mu + \mu'} \left(\underline{L}(T) - \underline{L}(0) \right) \tag{63}$$

be the displacement of the earth in time T.

(2) Set

$$\underline{R}'_{10} = \underline{R}_{10} - \underline{\Delta E} = \underline{R}_{10} + \frac{\mu'}{\mu + \mu'} \Delta \underline{L} \tag{64}$$

and

$$\dot{\underline{R}}'_{10} = \dot{\underline{R}}_{10} \,, \tag{65}$$

since a translation of the origin will not affect the velocity.

(3) Fix the moon at $\underline{L}$ (T), that is in its position at time T relative to the earth.

(4) Solve the two-fixed center problem with the moon (fixed at $\underline{L}$ (T)) and initial conditions $\underline{R}'_{10}$ and $\dot{\underline{R}}'_{10}$ to obtain an approximation at time T to the restricted problem with initial conditions $\underline{R}_{10}$ and $\dot{\underline{R}}_{10}$ and moon initially at $\underline{L}(0)$.

The analysis for a system rotating about any point other than the barycenter is carried out in a similar way, but the algebra is more complicated. The origin of the rotating system is to be the point A, defined by Eq. (4), with α and γ determined from Eqs. (58) and (60).

In Fig. 3, the vector A and the original and modified initial conditions are shown in the rotating system.

Again, it is seen that the two-fixed center problem, with primed initial conditions and unprimed positions of earth and moon, is related to that with unprimed initial conditions and primed positions of earth and moon by a rigid rotation which is the rotation part of the transformation carrying the rotating system into the fixed system. It must be remembered, however, that unlike the barycenter B, which may be regarded as a fixed inertial point, A is an accelerated point in inertial space, so that more than a rotation is required to transform back from the rotating system to the fixed system. In Fig. 4, the system rotating about A is shown at $t = 0$ and $t = T$.

It is now easy to see that the translation required to complete the transformation to axes moving with A, but with fixed directions, is a translation from A to A$'$. Actually, this translation need not be considered further because it is desired to find modification in the initial conditions relative to the earth rather than relative to A.

Referring again to Fig. 3, it is seen that the primed positions of the earth and the moon define a line parallel to that of the earth and moon at time T in the fixed system. Thus, just as in the barycenter case,

$$\underline{R}'_{10} = \underline{R}_{10} - \underline{\Delta E}$$

and (66)

$$\underline{P}' = \underline{P}.$$

To obtain $\underline{\Delta E}$, one may note that $\underline{\Delta E}$ is obtained by a rotation of E through ωT about A and that this $\underline{\Delta E}$ is just the negative of a rotation of A through ωT about E. The vector $\underline{A}$, relative to E, is given by

$$\underline{A}_E = \underline{A} + \frac{\mu'}{\mu + \mu'} \underline{L} = (\alpha + \frac{\mu'}{\mu + \mu'}) \underline{L} + \gamma \dot{\underline{L}}, \tag{67}$$

and the change in $\underline{A}_E$ induced by a rotation of $\underline{A}_E$ through ωT about E is given by

$$\underline{\Delta A}_E = (\alpha + \frac{\mu'}{\mu + \mu'}) (\underline{L}(T) - \underline{L}(0)) + \gamma (\dot{\underline{L}}(T) - \dot{\underline{L}}(0)) \tag{68}$$

$$= - \underline{\Delta E},$$

so that finally,

$$\underline{R}'_{10} = \underline{R}_{10} + (\alpha + \frac{\mu'}{\mu + \mu'}) (\underline{L}(T) - \underline{L}(0)) + \gamma (\dot{\underline{L}}(T) - \dot{\underline{L}}(0)). \tag{69}$$

As before, $\underline{P}$, which may now be regarded as $\dot{\underline{R}}_{10}$ in the fixed system, is unmodified. The two-fixed center problem, with $\underline{R}'_{10}$ and $\dot{\underline{R}}_{10}$ as initial conditions with the moon fixed at $\underline{L}$ (T) relative to the earth, should produce, at time T, a good approximation to the restricted problem, with initial condition $\underline{R}_{10}$ and $\dot{\underline{R}}_{10}$ and the moon initially at $\underline{L}$ (0), provided T is small enough so that the second and higher order time derivatives of J_2 produce a negligible effect.

PRELIMINARY NUMERICAL RESULTS

The parameters α and γ have been determined for a lunar orbit with the following initial conditions:

$$x_{10} = -37163.638 \text{ km}$$
$$y_{10} = -56452.867 \text{ km}$$
$$z_{10} = -30844.317 \text{ km}$$
$$\dot{x}_{10} = -0.65536162 \text{ km/sec}$$
$$\dot{y}_{10} = -2.7369109 \text{ km/sec}$$
$$\dot{z}_{10} = -1.0459904 \text{ km/sec}$$

The distance of the vehicle from the earth is about 11.6 earth radii, and it has a speed of about 3 km/sec. For these conditions, the values of α and γ are the following:

$$\alpha = -6.2611792 \times 10^{-4}$$
$$\gamma = 0.28110731 \text{ hr}$$

The two-fixed-center calculation with the initial conditions modified for evaluation of the position and velocity of the vehicle at 23, 33, and 53 hours was compared with the base orbit at 23, 33, and 53 hours respectively. The deviations in position of the two-fixed-center calculation from the base case are shown in the table below. Included in the same table are the deviations of the corresponding Kepler problem from the base case.

Time	Dist. from Earth	Deviation	Two-Fixed -Center	Kepler
23 hr	35.3 ER	Δx	144 km	170 km
		Δy	132 km	200 km
		Δz	33 km	10 km
33 hr	42.1 ER	Δx	262 km	430 km
		Δy	155 km	250 km
		Δz	142 km	30 km
53 hr	52.7 ER	Δx	1300 km	1970 km
		Δy	1080 km	1100 km
		Δz	993 km	110 km

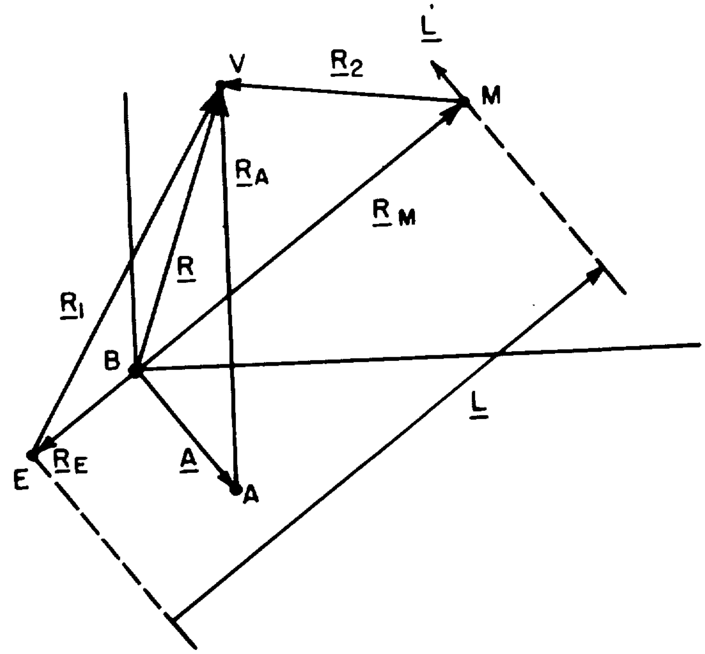
It can be seen from the table that the deviations resulting from the use of the two-fixed-center problem are slightly smaller than those of the Kepler problem. It is desirable to obtain much smaller deviations than these, but, because α and γ used are determined only from the initial conditions, one could hardly expect better results. The use of one of the more sophisticated methods for determining α and γ, outlined earlier, should lead to considerable improvement. As noted earlier, these methods would render α and γ dependent on time as well as on the initial conditions. The smallness of the deviations (all are under $\frac{1}{2}$%) indicates that times of at least to 60 hours could be used without prejudicing the validity of the approximation.

REFERENCES

1. Schulz-Arenstorff, R., Davidson, M.J., Jr., and Sperling, H.J., "The Restricted Three-Body-Problem As a Perturbation of Euler's Problem of Two-Fixed-Centers and Its Applications to Lunar Trajectories," Proceedings of the National Meeting on Manned Space Flight, Institute of Aerospace Sciences, 1962.

2. Arenstorf, R., Notes on the Restricted Problem in Two Dimensions (Private Communication).

3. Pines, S. and Payne, M., The Application of the Two-Fixed-Center Problem to Lunar Trajectories, Report RAC 484, Republic Aviation Corporation, 30 October 1961.



V
$\underline{R}_2$
$\underline{L}'$
M
$\underline{R}_A$
$\underline{R}_M$
$\underline{R}$
$\underline{R}_1$
B
$\underline{L}$
E
$\underline{R}_E$
$\underline{A}$
A

FIGURE 1

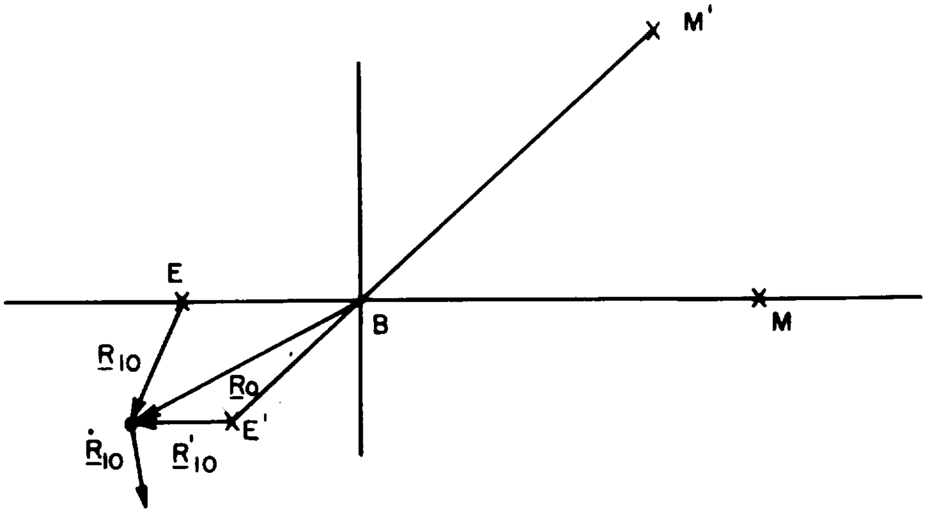

M'
E
B
M
$\underline{R}_{10}$
$\underline{R}_0$
E'
$\dot{\underline{R}}_{10}$
$\underline{R}'_{10}$

FIGURE 2

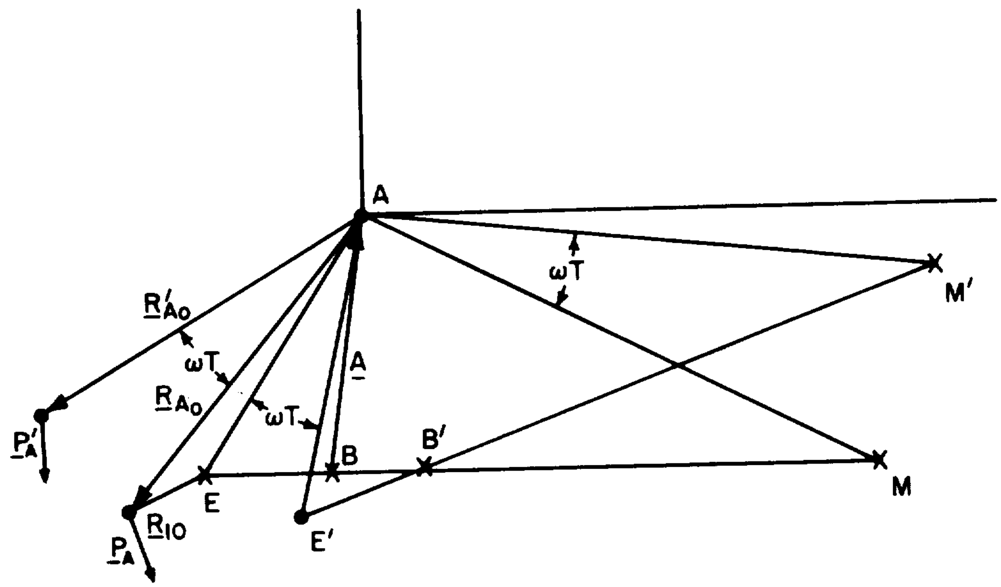

A
ωT
M′
$\underline{R}'_{A0}$
ωT
$\underline{A}$
$\underline{R}_{A0}$
ωT
$\underline{P}'_A$
B
B′
E
M
$\underline{R}_{10}$
E′
$\underline{P}_A$

FIGURE 3

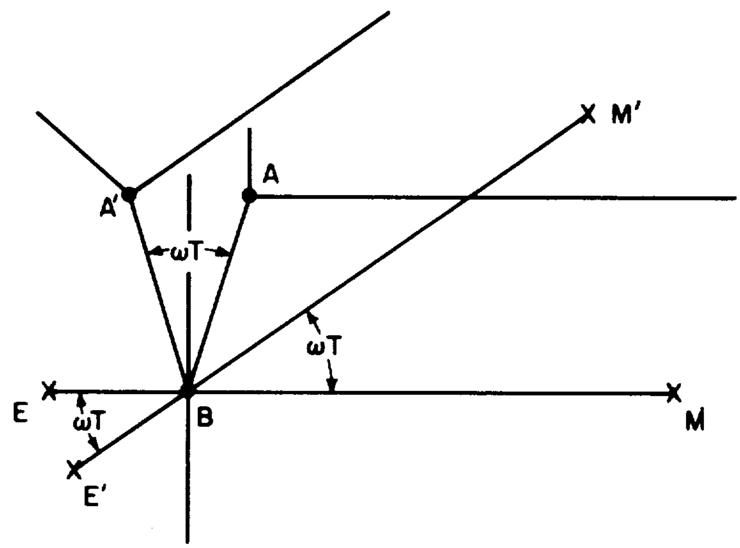


FIGURE 4

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A Recursion Process for the Generation of Orthogonal Polynomials in Several Variables

by

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SUMMARY

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A recursive process for generating multi-variable orthogonal polynomials is developed.

I. INTRODUCTION

Let $[\beta_0, X(\beta_0)]$, $[\beta_1, X(\beta_1)]$, ..., $[\beta_n, X(\beta_n)]$ be a collection of tabular points, where $\beta = (t_0, t_1, \ldots, t_m)$ and $\beta_i = (t_{i0}, t_{i1}, \ldots, t_{im})$. That is,

$$\beta_0 = (t_{00}, t_{01}, \ldots, t_{0m})$$
$$\beta_1 = (t_{10}, t_{11}, \ldots, t_{1m})$$
$$\vdots$$
$$\beta_n = (t_{n0}, t_{n1}, \ldots, t_{nm}),$$

where the first subscript denotes the particular point and the second subscript denotes the particular variable. The least squares problem for several variables is that of finding a polynomial

$$A_0\varphi_0(\beta) + A_1\varphi_1(\beta) + \ldots + A_N\varphi_N(\beta) = \sum_{j=0}^{N} A_j\varphi_j(\beta)$$

such that

$$\sum_{i=0}^{n} [X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i)]^2$$

is a minimum. Sufficient conditions for the existence of this polynomial were developed in [1].

Now suppose we let

$$F(A_0, A_1, \ldots, A_N) = \sum_{i=0}^{n} [X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i)]^2.$$

A necessary condition that this be a minimum is that $\frac{\partial F}{\partial A_0} = \frac{\partial F}{\partial A_1} = \ldots = \frac{\partial F}{\partial A_N} = 0$. This yields the system of equations

$$A_0\overline{\varphi}_0\cdot\overline{\varphi}_0 + A_1\overline{\varphi}_1\cdot\overline{\varphi}_0 + \ldots + A_N\overline{\varphi}_N\cdot\overline{\varphi}_0 = \overline{X}\cdot\overline{\varphi}_0$$
$$A_0\overline{\varphi}_0\cdot\overline{\varphi}_1 + A_1\overline{\varphi}_1\cdot\overline{\varphi}_1 + \ldots + A_N\overline{\varphi}_N\cdot\overline{\varphi}_1 = \overline{X}\cdot\overline{\varphi}_1$$
$$\vdots$$
$$A_0\overline{\varphi}_0\cdot\overline{\varphi}_N + A_1\overline{\varphi}_1\cdot\overline{\varphi}_N + \ldots + A_N\overline{\varphi}_N\cdot\overline{\varphi}_N = \overline{X}\cdot\overline{\varphi}_N \;;$$

where $\overline{X} = [X(\beta_0),X(\beta_1),\ldots,X(\beta_n)]$ and $\overline{\varphi}_j = [\varphi_j(\beta_0),\varphi_j(\beta_1),\ldots,\varphi_j(\beta_n)]$, $j = 0,1,\ldots,N$. Obviously, if $\varphi_0(\beta)$, $\varphi_1(\beta)$, ..., $\varphi_N(\beta)$ are chosen so that $\overline{\varphi}_i \cdot \overline{\varphi}_j = \delta_{ij}$, then the problem is greatly simplified.

II. The Recursion Process

Let $\overline{g}_0$, $\overline{g}_1$, ..., $\overline{g}_m$ be the following set of vectors:

$$\overline{g}_0 = (t_{00},t_{10},\ldots,t_{n0})$$

$$\overline{g}_1 = (t_{01},t_{11},\ldots,t_{n1})$$

$$\vdots$$

$$\overline{g}_m = (t_{0m},t_{1m},\ldots,t_{nm}),$$

and define the vectors $\overline{g}'_0$, $\overline{g}'_1$, ..., $\overline{g}'_m$ as follows:

$$\overline{g}'_\gamma = \overline{g}_\gamma - (\overline{g}_\gamma,\overline{e}_0)\overline{e}_0 - (\overline{g}_\gamma,\overline{e}_1)\overline{e}_1 - \ldots - (\overline{g}_\gamma,\overline{e}_{\gamma-1})\overline{e}_{\gamma-1},$$

$\gamma = 0,1,\ldots,m$, where $\overline{e}_\gamma = \overline{g}'_\gamma/\|\overline{g}'_\gamma\|$ and $(\overline{g}_\gamma,\overline{e}_k) = \overline{g}_\gamma \cdot \overline{e}_k$, $k = 0,1,\ldots,m-1$. Then the vectors $\overline{e}_0$, $\overline{e}_1$, ..., $\overline{e}_m$ form an orthonormal collection. Notice that

$$\overline{e}_k = 1/\|\overline{g}'_k\| [\overline{g}_k - (\overline{g}_k,\overline{e}_0)\overline{e}_0 - (\overline{g}_k,\overline{e}_1)\overline{e}_1 - \ldots - (\overline{g}_k,\overline{e}_{k-1})\overline{e}_{k-1}].$$

<u>Theorem</u>: If $A_\gamma(k) = (\overline{g}_\gamma,\overline{e}_k)/\|\overline{g}'_\gamma\|$, then

$$A_\gamma(k) = [A_\gamma(-1)A_k(-1)(\overline{g}_\gamma,\overline{g}_k) - A_\gamma(0)A_k(0) - A_\gamma(1)A_k(1) - \ldots$$

$$\ldots - A_\gamma(k-1)A_k(k-1)], \text{ for } k = 0,1,\ldots,\gamma-1.$$

<u>Proof</u>: $A_\gamma(k) = (\bar{g}_\gamma, \bar{e}_k)/\|\bar{g}'_\gamma\| = 1/\|\bar{g}'_\gamma\| [\bar{g}_\gamma, \bar{g}_k/\|\bar{g}'_k\| - (\bar{g}_k, \bar{e}_0)\bar{e}_0/\|\bar{g}'_k\| - \cdots$

$$\cdots - (\bar{g}_k, \bar{e}_{k-1})\bar{e}_{k-1}/\|\bar{g}'_k\|]$$

$$= (\bar{g}_\gamma, \bar{g}_k)/[\|\bar{g}'_\gamma\|\,\|\bar{g}'_k\|] - (\bar{g}_k, \bar{e}_0)(\bar{g}_\gamma, \bar{e}_0)/[\|\bar{g}'_k\|\,\|\bar{g}'_\gamma\|] - \cdots$$

$$\cdots - (\bar{g}_k, \bar{e}_{k-1})(\bar{g}_\gamma, \bar{e}_{k-1})/[\|\bar{g}'_k\|\,\|\bar{g}'_\gamma\|]$$

$$= A_\gamma(-1)A_k(-1)(\bar{g}_\gamma, \bar{g}_k) - A_\gamma(0)A_k(0) - \cdots - A_\gamma(k-1)A_k(k-1)$$

Thus,

$$\bar{e}_\gamma = A_\gamma(-1)\bar{g}_\gamma - A_\gamma(0)\bar{e}_0 - \cdots - A_\gamma(\gamma-1)\bar{e}_{\gamma-1}.$$

This theorem allows us to construct the following triangular array of coefficients that will be needed in later calculations:

$$\begin{array}{llll}
A_0(-1) & & & \\
A_1(-1) & A_1(0) & & \\
A_2(-1) & A_2(0) & A_2(1) & \\
A_3(-1) & A_3(0) & A_3(1) & A_3(2) \\
\cdots & \cdots & \cdots & \cdots
\end{array}$$

Notice that only the elements in the first column require any new calculations, since all other elements in the array can be written recursively using these elements and the previous theorem.

To show how these coefficients are to be utilized, let us define $f_\gamma(\beta)$ as follows:

$$f_\gamma(\beta) = A_\gamma(-1)t_\gamma - A_\gamma(0)f_0(\beta) - A_\gamma(1)f_1(\beta) - \dots - A_\gamma(\gamma-1)f_{\gamma-1}(\beta),$$

for $\gamma = 1,2,\dots,m$, and

$$f_0(\beta) = A_0(-1)t_0.$$

Notice that each $f_\gamma(\beta)$ is a linear combination of the $m + 1$ independent variables $t_0, t_1, \dots, t_m$.

<u>Theorem</u>: If $\overline{f}_\gamma = [f_\gamma(\beta_0), f_\gamma(\beta_1), \dots, f_\gamma(\beta_n)]$, $\gamma = 0,1,\dots,m$, then $\overline{f}_\gamma = \overline{e}_\gamma$.

<u>Proof</u>:
$$\begin{aligned}\overline{f}_0 &= [f_0(\beta_0), f_0(\beta_1), \dots, f_0(\beta_n)]\\ &= [A_0(-1)t_{00}, A_0(-1)t_{10}, \dots, A_0(-1)t_{n0}]\\ &= A_0(-1)[t_{00}, t_{10}, \dots, t_{n0}] = A_0(-1)\overline{g}_0 = \overline{e}_0.\end{aligned}$$

Now assume that $\overline{f}_{k-1} = \overline{e}_{k-1}$. Then

$$\begin{aligned}\overline{f}_k &= [f_k(\beta_0), f_k(\beta_1), \dots, f_k(\beta_n)]\\ &= [A_k(-1)t_{0k} - A_k(0)f_0(\beta_0) - A_k(1)f_1(\beta_0) - \dots - A_k(k-1)f_{k-1}(\beta_0);\\ &\quad A_k(-1)t_{1k} - A_k(0)f_0(\beta_1) - A_k(1)f_1(\beta_1) - \dots - A_k(k-1)f_{k-1}(\beta_1),\\ &\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\ &\quad A_k(-1)t_{nk} - A_k(0)f_0(\beta_n) - A_k(1)f_1(\beta_n) - \dots - A_k(k-1)f_{k-1}(\beta_n)]\end{aligned}$$

$$= A_k(-1)[t_{0k}, t_{1k}, \ldots, t_{nk}] - A_k(0)[f_0(\beta_0), f_0(\beta_1), \ldots, f_0(\beta_n)]$$

$$- A_k(1)[f_1(\beta_0), f_1(\beta_1), \ldots, f_1(\beta_n)] - \cdots\cdots$$

$$- A_k(k-1)[f_{k-1}(\beta_0), f_{k-1}(\beta_1), \ldots, f_{k-1}(\beta_n)]$$

$$= A_k(-1)\overline{g}_k - A_k(0)\overline{f}_0 - A_k(1)\overline{f}_1 - \cdots - A_k(k-1)\overline{f}_{k-1}$$

$$= A_k(-1)\overline{g}_k - A_k(0)\overline{e}_0 - A_k(1)\overline{e}_1 - \cdots - A_k(k-1)\overline{e}_{k-1} = \overline{e}_k.$$

Thus, if we take $f_j(\beta) = \varphi_j(\beta)$, $j = 0,1,\ldots,N$, in the normal equations, then the solution is

$$A_0 = \overline{X}\cdot\overline{f}_0$$
$$A_1 = \overline{X}\cdot\overline{f}_1$$
$$\vdots$$
$$A_N = \overline{X}\cdot f_N,$$

where we must have $N = m$. Therefore, the approximating function is

$$A_0t_0 + A_1t_1 + \cdots + A_mt_m.$$

Now suppose we desire an approximating polynomial containing a constant term as well as all possible second degree terms. Then we will need to include the following vectors in this treatment:

$$\overline{g}_{m+1} = (1,1,\ldots,1)$$
$$\overline{g}_{m+2} = (t_{00}^2, t_{10}^2, \ldots, t_{n0}^2)$$
$$\overline{g}_{m+3} = (t_{01}^2, t_{11}^2, \ldots, t_{n1}^2), \text{ etc.,}$$

denote typical elements of these vectors by the variables t_{m+1}, t_{m+2}, t_{m+3}, etc., and proceed as before in deriving the A_γ coefficients and the function $f_\gamma(\beta)$.

BIBLIOGRAPHY

1. Dupree, Daniel E., Existence of Multivariable Least Squares Approximating Polynomials, Progress Report No. 2 on Studies in the Fields of Space Flight and Guidance Theory.

2. Achieser, N. I., Theory of Approximation, Ungar Publishing Company.

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NUMERICAL APPROXIMATION OF MULTIVARIATE FUNCTIONS APPLIED TO THE ADAPTIVE GUIDANCE MODE (PART II)

by

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NUMERICAL APPROXIMATION OF MULTIVARIATE FUNCTIONS APPLIED TO THE ADAPTIVE GUIDANCE MODE (PART II)

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Summary

A procedure using quadrature methods and combinatorial topology is described for computing values of integrals in n-dimensions. This offers one way of solving the problem of data point selection for the generation of a least squares approximation of a multivariate function by a linear combination of polynomials which are orthonormal over a region.

SECTION I. INTRODUCTION

Progress Report No. 2 on Studies in the Fields of Space Flight and Guidance Theory contained the first part of this investigation of the approximation of functions χ and T_R of many variables using a least squares criterion. An iterative method of generating a family of orthonormal functions $\{q_i\}$ was described. This method is now part of a computer program for approximating a function given in the form of tabulated data. Generalized Fourier coefficients c_i are formed using the definition

$$c_i = (q_i, \chi) = \sum_{k=1}^{n} w_k q_{ik} \chi_k . \tag{1}$$

where (q_i, χ) is the inner product of the function q_i with the control function χ, w_k is a weight applied to the k^{th} data point. The two guidance functions χ and T_R were then approximated

$$\chi \approx \sum_{j=1}^{r} (\chi, q_j) q_j \tag{2}$$

$$T_R \approx \sum_{j=1}^{r'} (T_R, Q_j) Q_j \tag{3}$$

The methods of selection of the weights w_i and the n data points were not discussed extensively. This aspect of the solution will now be presented.

SECTION II. THE GENERALIZED FOURIER COEFFICIENTS, ORTHONORMALIZATION, AND MULTIPLE INTEGRALS.

Both the orthonormalization of a given set of basis functions $\{b_i\}$ to form the set $\{q_i\}$ or $\{Q_i\}$ and the formation of the generalized Fourier coefficients c_i depend on the definition of the inner product (b_i, b_j) of two functions.

Usually there are uncountably many minimum fuel trajectories which fulfill a given mission if the initial conditions lie within some closed bounded region. These trajectories form a region R over which the functions $T_R\ (t, \frac{F}{W}, x, \dot{x}, y, \dot{y}, \frac{\dot{m}}{m})$ and $\chi(t, \frac{F}{W}, x, \dot{x}, y, \dot{y}, \frac{\dot{m}}{m})$ are defined. This region R has bounds imposed by the physical aspects of the problem or by restrictions on the initial values. A true least squares approximation of χ

requires that

$$(\chi, q_i) = \int_R \chi q_i dR$$

$$= \int \cdot \cdot \cdot \int_R \chi q_i dx d\dot{x} dy d\dot{y} d(\tfrac{F}{W}) d(\tfrac{\dot{m}}{m}) dt$$

Inner products (T_R, Q_i) are similarly defined. The accuracy of the approximation depends on the accuracy with which the numerical values of the multiple integrals are determined. At first consideration, the computation of these multiple integrals seems a problem of at least the same magnitude as the original one of approximating a multivariate function. This is easily seen since

$$\int \cdot \cdot \cdot \int_R f(x_1, x_2 \ldots x_r)\, dx_1 dx_2 \ldots dx_r = \sum_{i=1}^{k} w_i f(x_{1i}, x_{2i} \ldots x_{ri}) - E(f) \qquad (4)$$

If the error E(f) is zero for polynomials of degree d or less in the r variables, then the quadrature formula is said to be of degree d. In this case, the direct way to obtain the set of weights, w_i, and points $(x_{1i}, x_{2i}, \ldots x_{ri})$ would be to solve the non-linear, non-homogeneous, algebraic equations obtained from (4) by substitution of a monomial for f.

$$\sum_{i=1}^{k} w_i \prod_{j=1}^{r} (x_{ji})^{dj} = \int \cdot \cdot \cdot \int_R \prod_{j=1}^{r} (x_j)^{dj} dx_j \qquad (5)$$

for all sets of d_j such that $\sum d_j \leq d$. This would involve solving a system of $\frac{(d+r)!}{d!r!}$ such equations. One solution would result in a quadrature formula.

This direct method, unfortunately, leads to more complicated problems than the original one of approximating inner products. However, we can now use both classical and "modern" developments of mathematics to provide alternate methods of evaluating these inner products. With only a sample of trajectories, the problem of point and weight selection for equations 1 - 3 can be reduced by special methods to finding quadrature formulae for the simplest geometric figures (simplexes) in a finite dimensional space.

SECTION III. SIMPLEXES, MULTIPLE INTEGRALS, AND QUADRATURE METHODS.

A set of points $p_o, p_1, \ldots, p_r$ in r-dimensional Euclidean space E^r is said to be linearly independent if the set of vectors (or elements) (p_1-p_o), $(p_2-p_o), \ldots, (p_r-p_o)$ are linearly independent; i.e., if

$$\alpha_1(p_1-p_o) + \alpha_2(p_2-p_o) + \ldots + \alpha_r(p_r-p_o) = 0$$

implies $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$ where the α_i are real numbers. If $v_o, v_i, \ldots, v_r$ are independent points, then the set of points p^* of the form

$$p^* = \alpha_o v_o + \alpha_1 v_1 + \ldots + \alpha_r v_r$$

where

$$\sum_{i=0}^{r} \alpha_i = 1$$

and

$$\alpha_i \geq 0, \; i = 0, 1, \ldots, r$$

is called a simplex with vertices v_i. For a given point p in the simplex, the α_i are called the barycentric coordinates of p.

Any simplex which has S ($S \leq r$) of the v_i as vertices is a proper face of the original r-dimensional simplex. Two r-dimensional simplexes are properly situated if their intersection is a common proper face or the null set θ.

A finite set of properly situated simplexes is called a complex. The set of minimum fuel trajectories representing possible disturbances in the state of the vehicle and still completing the mission are given as tabulated data. Along each trajectory, the values of the parameters are given as 7-tuples with the associated values of the two control functions χ and T_R. These 7-tuples define the region R for the purpose of a least squares approximation using numerical methods. The complete set of tabulated trajectories defines a complex. This complex C is an approximation of the region R.

By decomposing the complex C into properly situated simplexes, integrating over each simplex, and finally summing the values of the multiple integral over each simplex, an approximation of the integral over the region R is obtained. In our particular case, we wish to find weights w_i and points $p_i = (t_i, \frac{F}{W}, x_i, \dot{x}_i, y_i, \dot{y}_i, \frac{\dot{m}}{m}_i)$ such that

$$\int_{S_7} P_d dv = \sum_{i=1}^{n} w_i P_d(t_i, \frac{F}{W}_i, x_i, \dot{x}_i, y_i, \dot{y}_i, \frac{\dot{m}}{m}_i)$$

where S_7 is a 7-dimensional simplex and P_d is a polynomial of degree less than or equal to d in the seven variables.

Integration over each simplex S_i can be an iterative task for a computer by the use of one quadrature formula of a given degree d. The number of points at which a function must be evaluated for use in a (2m-1) order quadrature formula that is valid over an r-simplex is equal to m^r in most cases. For example, with the immediate problem at hand, if we wished to integrate a 5^{th} degree polynomial in 7 variables exactly over a 7-simplex, $3^7 (=2187)$ tabulated points would be needed for each simplex. If the complex C were composed of only a few simplexes, the use of such elaborate formulae could possibly be justified in terms of computer time taken and the accuracy of the results obtained. However, the parameters and the control functions are reasonably smooth, indicating that simpler quadrature formulae requiring fewer data points may be used. This would be especially true if the euclidean distances between the vertices v_i of the simplex S_r are small and the values of the parameters do not change rapidly within the simplex.

Recent work by Stroud (Ref. 6) would indicate that it is possible to find formulas requiring far fewer than m^r points for (2m-1)-degree integration. A third degree formula for a r-dimensional simplex was developed using only 2r+3 points and not 2^r points. Unfortunately, at this time, there seems to be no general theory for the generation of these simpler quadrature methods.

The hypervolume Δr of a simplex S_r with vertices $v_i = (x_{i1}, x_{i2}, \ldots x_{ir})$ is required in the development of quadrature formulas. This hypervolume is easily computed in the form of the absolute value of a determinant

$$\Delta r = \frac{1}{r!} \left\| \begin{matrix} 1 & x_{o1} & x_{o2} & \cdots & x_{or} \\ 1 & x_{11} & x_{12} & \cdots & x_{1r} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & x_{r1} & x_{r2} & \cdots & x_{rr} \end{matrix} \right\|$$

For an extensive explanation of this formula see Ref. 5. This same determinant may be used to test the independence of the vertices (points) v_i, $i = 0, 1, \ldots r$.

Quadrature formulas may be developed to give the exact value of the integral of a degree d polynomial in r variables over a r-simplex S_r. (See Ref. 1, 2, 3, 6, 7, and 8.) However, for computer use, an affinely symmetric formula is desirable. In this type of formula, the weight w_i for the point p_i does not change when the r-dimensional space containing the simplex is affinely transformed. In other words, if w_i is the weight associated with the point p_i in the simplex S_r, then w_i is associated with the point

$$\underline{p}_i^* = \underline{A}\,\underline{p}_i + \underline{\gamma} \tag{6}$$

$$= T(p_i) \tag{7}$$

where the points $\underline{p}_i$ and $\underline{p}_i^*$ are written as column vectors, A is a non-singular matrix of real coefficients, and γ is a column of constants. Equation (7) is (6) written with the affine operator T. p_i^* is in the simplex $S_r^* = T(S_r)$, the set of transformed points of S_r. An additional requirement may be imposed on a quadrature formula for a simplex. The points p_i used in the formula all must lie within the given simplex. This restriction is justified for two reasons:

(1) The function may not exist outside the simplex.

(2) Since any point within a simplex is determined by its barycentric coordinates, a simple computer routine can be used to find the quadrature points from the vertices of the simplex.

An affinely symmetric formula of third degree can now be given. Let the

$r+1$ vertices of the r-simplex S_r be v_o, v_1, . . ., v_r. The barycenter B of S_r is defined by

$$B = \frac{1}{r+1} \sum_{o}^{r} v_i .$$

The hypervolume of S_r is Δr. Then

$$\int_{S_r} f dv_r = a_r \sum_{o}^{r} f(u_i) + c_r f(B)$$

where

$$a_r = \frac{(r+3)^2}{4(r+1)(r+2)} \Delta r$$

$$c_r = \frac{-(r+1)^2}{4(r+2)} \Delta r$$

$$u_i = \frac{2}{r+3} v_i + \frac{r+1}{r+3} B \quad i = 0, 1, \ldots, r$$

The formula is exact when f is a 3rd degree polynomial in r variables. This means that the values of inner products of the types (x_i^m, x_j^n), $(m+n = 3;\ i, j = 1, \ldots r)$ will be exact over each simplex S_r in the region R. If the inner product is of the type $(x^i, f), i = 0, 1, 2,$ where f is not a polynomial of degree 3 or less, then there will be an error due to the quadrature formula. There has been little error analysis available for quadratures involving functions of many variables. However, the errors arising by using such a definite procedure as the above are usually much less than if a simple sum of products had been used to approximate an inner product.

The approximation by the use of quadrature formulas is an approximation over an entire region R and not over a finite set of points as in a least squares method such as normal equations or orthonormalization of vectors.

SECTION IV. RECOMMENDATIONS

In order to arrive at a suitable general algorithm for the approximation of control functions for Saturn class vehicles, it is recommended that the studies contained in this report and Part I be continued. This continuation should include the following particular areas of effort:

1. The development of a method, suitable for computers, for finding the vertices of all the properly situated simplexes in the region defined by the tabulated minimum fuel trajectories.
2. The implementation of available information from the calculus of variations and multivariate functions to determine the boundary of the region over which minimum fuel trajectories are defined for a particular mission.
3. The comparison of the accuracy of an approximation using quadrature methods to define the inner product of two functions with the usual method using sums of products of the values of the two functions at arbitrary points.
4. The study of possible methods of directly producing a rational approximation from a polynomial approximation or a partial sum of a series.
5. The investigation of direct substitution of a polynomial with undetermined coefficients for the control functions into the Euler-Lagrange equations; the goal being the determination of the coefficients which will minimize the fuel consumption for a particular mission. A generalization of the two point boundary problem would be needed with inequality constraints or "initial conditions" satisfying some inequality.
6. The exploitation of analog computer methods, which may possess advantages in terms of time and money, in the areas of both Chybechev and least squares approximations deserves renewed effort.

SECTION V. REFERENCES

1. P. C. Hammer, O. J. Marlowe, A. H. Stroud, "Numerical Integration over Simplexes and Cones". Math Tables and Aids to Comp., Vol. 10, No. 55 (1956) pp 130-137.

2. P. C. Hammer, A. H. Stroud, "Numerical Integration over Simplexes" Math Tables and Aids to Comp., Vol. 10, No. 55 (1956) pp 137-139.

3. P. C. Hammer, A. W. Wymore, "Numerical Evaluation of Multiple Integrals I". Math Tables and Aids to Comp., Vol. 11, No. 57 pp 59-97.

4. L. S. Pontryagin, <u>Foundations of Combinatorial Topology</u>, Graylock Press, Rochester, N. Y., 1952.

5. D. M. Y. Sommerville, <u>An Introduction to the Geometry of N Dimensions</u>, Dover Pub., New York, 1958.

6. A. H. Stroud, "Numerical Integration Formulas of Degree 3 for Product Regions and Cones". Math. of Comp., Vol. 15, No. 74 (1961) pp 143-150.

7. H. C. Thatcher, "Optimum Quadrature Formulas in S Dimensions, Math Tables and Aids to Comp., Vol. 11, No. 59 (1957) pp 189-194.

8. G. W. Tyler, "Numerical Integration of Functions of Several Variables", Canadian J. Math, Vol. 5 (1953) pp 393-412.

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THE APPLICATION OF LINEAR PROGRAMMING TO MULTIVARIATE APPROXIMATION PROBLEMS

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SUMMARY

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The purpose of this work is to obtain simple, formal functions which approximate, in some sense, the steering and cutoff functions derived in the Adaptive Guidance Mode. The approach taken in this report is to use linear programming techniques to fit linear combinations of known functions or ratios of such functions to a set of tabulated values of the steering and cutoff functions.

I. INTRODUCTION

This report describes the use of linear programming techniques to approximate the steering and cutoff functions for the implementation of the Adaptive Guidance Mode. [5] [8] [10] This approach to the approximation of the guidance functions is basically a multivariate curve-fitting problem. Values of the steering and cutoff functions are tabulated for a representative set of points on minimum fuel trajectories and then, formal functions are sought which approximate these tabulated functions according to some criterion. When this criterion is L_1 (minimized sum of absolute deviations) or L_∞ (Chebyshev or minimized maximum deviation), linear programming may be used to determine the approximating functions. An analysis has been made of the case in which the approximating functions are polynomials. Studies have been initiated on the use of ratios of polynomials for the approximating functions.

Section II of the report contains some results in the theory of linear programming which are included as background for the later discussions.

Section III contains a statement of the curve-fitting problem.

In section IV, the case of the L_1 approximation of the guidance functions by a polynomial is considered. This case is included for completeness and for purposes of comparison with the L_∞ case, which is of more interest in practical applications.

The L_∞ approximation of the guidance functions by a polynomial is considered in section V. The problem is also formulated so that a linear programming routine can be used to find that polynomial, if it exists, which approximates the tabulated function to within predetermined tolerances at each data point.

Section VI contains a discussion of peculiarities of the curve-fitting problem which cause slow convergence of the linear programming method. Recommendations for improving this speed of convergence are included.

Numerical examples of the L_1 and L_∞ approximation of the steering function by polynomials are given in section VII.

Section VIII contains a brief discussion of experiments done using alternative methods for choosing the pivotal elements in the simplex algorithm for linear programming. The purpose of this work was a further increase in the speed of convergence for the simplex method.

II. THEORETICAL BACKGROUND

Linear programming problems which arise in curve-fitting can often be solved more readily in their dual form than in the original primal form. The basic properties of dual linear programming are therefore summarized in this section.

The Duality Theorem states that if the primal (dual) problem has a finite optimum solution, then the dual (primal) problem has a finite optimum solution and the extrema of the respective objective functions are equal. If the primal (dual) problem has an unbounded optimum solution, then the dual (primal) problem has no feasible solutions.

If a bounded optimum solution for the primal problem exists, then the solution of the dual problem can be obtained by solving the primal problem, and vice versa. The desired solution can therefore best be obtained by solving the simplest of the primal and dual problems.

There are several pairs of dual linear programs. [1] [2] [3] The most familiar pair consists of the "canonical" or "symmetric" dual programs. Goldman and Tucker [3] point out that the other problem pairs are essentially no more general than this canonical one. Several pairs of dual linear programs are discussed below. In practice, the appropriate pair must be chosen to fit the special requirements of the particular curve-fitting problem.

(A) The canonical pair of dual linear programs is stated as follows:

Primal problem (dual problem):

(1) Minimize the objective function, $f = CX$, subject to the constraints, $AX \geq b$ and $X \geq 0$, where X is an n-component column vector of unknowns, C is an n-component row vector, A is an $m \times n$ matrix, and b is an m-component column vector.

Dual problem (primal problem):

(2) Maximize the objective function, $g = Wb$, subject to the constraints, $WA \leq C$ and $W \geq 0$, where W is an m-component row vector of unknowns and A, C, and b are as defined in (1).

To solve (1), the constraints, $AX \geq b$, are converted to equalities of the form, $(A\ ,\ -1)\begin{pmatrix} X \\ \lambda \end{pmatrix} = b$, where λ is an m-component column vector of non-negative "slack" variables and I is the $m \times m$ identity matrix. The linear programming problem is then stated as,

(3) Minimize the objective function, $f = CX$, subject to the constraints,

$$(A\ ,\ -I)\begin{pmatrix} X \\ \lambda \end{pmatrix} = b\ ,\quad X \geq 0\ ,\quad \lambda \geq 0\ .$$

To show that the solution of (2) can be obtained by solving (3), we let X_0 and W_0 denote the optimum solutions of the primal and dual problems, respectively. Let B denote the optimum basis of $(A\ ,\ -I)$. X_0 is an m-component, column vector of basis variables (its elements correspond to those columns of $(A\ ,\ -I)$ which occur in the basis B). Now let C_0 denote an m-component, row vector, each element of which is an element of C corresponding to a basis variable. Consider the vector $W_0 = C_0 B^{-1}$. $B^{-1}A$ is an $m \times n$ matrix, each element of which is an element in the simplex tableau for the solution of (3), and X_0 is the optimum solution of (3); therefore all the shadow prices are non-negative. Hence, $C - C_0 B^{-1} A \geq 0$ and $0 - C_0 B^{-1}(-I) \geq 0$. Therefore $W_0 A \leq C$ and $W_0 \geq 0$. Hence W_0 is a feasible solution of (2). Furthermore, $BX_0 = b$; therefore W_0 satisfies $C_0 X_0 = W_0 b$. It then follows from the Duality Theorem that W_0 is the optimum feasible solution of (2).

(B) The unsymmetric pair of dual programming problems can be stated as follows:

Primal problem (dual problem):

(4) Minimize the objective function, $f = CX$, subject to the constraints $AX = b$ and $X \geq 0$.

Dual problem (primal problem):

(5) Maximize the objective function, $g = Wb$, subject to the constraints, $WA \leq C$, where W is unrestricted in sign.

This pair of problems can be obtained from (1) and (2) by expressing the equality constraint $AX = b$ as a pair of inequality constraints, $AX \geq b$ and $-AX \geq -b$. By a proof similar to that for the canonical pair of dual problems it can be shown that the solution $C_0 B^{-1}$ of (5) can be obtained by solving (4), where C_0 and B are as previously defined.

(C) The pair of dual problems for a linear programming problem with mixed constraints can be stated as follows:

Primal problem (dual problem):

(6) Maximize the objective function, $f = C_1 X_1 + C_2 X_2$, subject to the constraints,
$A_{11} X_1 + A_{12} X_2 \leq b_1$, $A_{21} X_1 + A_{22} X_2 = b_2$, $X_1 \geq 0$ and X_2 unrestricted in sign.

Dual problem (primal problem):

(7) Minimize the objective function, $g = W_1 b_1 + W_2 b_2$, subject to the constraints,
$W_1 A_{11} + W_2 A_{21} \geq C_1$, $W_1 A_{12} + W_2 A_{22} = C_2$, $W_1 \geq 0$ and W_2 unrestricted in sign.

The solution, $C_0 B^{-1}$, of (7) can again be obtained by solving (6).

III. THE CURVE-FITTING PROBLEM

The purpose of this work is to obtain simple, formal functions which approximate, in some sense, the steering and cutoff functions for a missile on a minimum fuel trajectory. The approach taken in this report is to use linear programming techniques to fit linear combinations of known functions or ratios of such functions to a set of tabulated values of the steering and cutoff functions. The studies were performed for a flat, two-dimensional earth with no atmosphere, a point missile, and constant fuel flow. The steering function, $\varkappa$, and the cutoff function, T , are functions of the six independent variables, x and y (rectangular space-fixed position coordinates), $\dot{x}$ and $\dot{y}$ (velocity coordinates), F/m (thrust acceleration), and t (time).

The approximation of the steering and cutoff functions is studied by considering the general problem of approximating a function, $f(\vec{z})$, whose value is known at n points, $\vec{z}_1 , \ldots, \vec{z}_n$, in an m-dimensional space, by a function, $P(\vec{z})$, of a given form in the components of the vector $\vec{z}$. When the criterion of "best fit" of $P(\vec{z})$ to $f(\vec{z})$ is L_1 , i.e., the sum of the absolute deviations,

$$\sum_{k=1}^{n} |P(\vec{z}_k) - f(\vec{z}_k)| ,$$ is minimized, or L_∞ , i.e., the maximum absolute deviation, $$\max_{k=1,1;n} |P(\vec{z}_k) - f(\vec{z}_k)| ,$$ is minimized, then the function $P(\vec{z})$ can be determined by linear programming techniques.

The function $P(\vec{z})$ is assumed to be of the form,

$$P(\vec{z}_k) = A_0 + \sum_{i=1}^{m} A_i P_i(z_{ki}) + \sum_{i>j=1}^{m} A_{ij} P_{ij}(z_{ki}, z_{kj})$$

$$+ \sum_{h>i>j=1}^{m} A_{hij} P_{hij}(z_{kh}, z_{ki}, z_{kj}) + \ldots,$$

where the A_i's , A_{ij}'s , ... are unknown coefficients to be determined, z_{ki} is the i-th component of $\vec{z}$ at the k-th data point, and $P_i(z_{ki})$, $P_{ij}(z_{ki}, z_{kj})$, ... are predetermined functions of z_i , $(z_i z_j)$, ..., respectively. In sections IV-VII, $P(\vec{z})$ is a polynomial, i.e., $P_i(z_{ki}) = z_{ki}$, $P_{ij}(z_{ki}, z_{kj}) = z_{ki} z_{kj}$, etc.

Various pairs of dual linear programming problems are formulated in sections IV and V for the curve-fitting problem. The number of constraints (other than those requiring variables to be non-negative) in one problem of the pair is usually of the order of n , the number of data points. The number of constraints for the other problem of the pair is of the order of the number of unknown coefficients in $P(\vec{z})$. In practice, n is very much larger than the number of coefficients. The time required to compute the optimum solution is approximately proportional to the cube of the number of constraints. Hence, a significant decrease in computation time can be obtained by solving that problem of the pair of dual problems in which the number of constraints is a function of the number of unknown coefficients in $P(\vec{z})$.

IV. THE L_1 APPROXIMATION

In an L_1 approximation, the function, $P(\vec{z})$, which minimizes the sum of absolute deviations, $\sum_{k=1}^{n} |P(\vec{z}_k) - f(\vec{z}_k)|$, is sought. For simplicity, only functions of the form $P(\vec{z}_k) = A_0 + \sum_{i=1}^{m} A_i z_{ki}$ will be considered.

The curve-fitting problem can be restated as the following linear programming problem. [11]

$$
(8)\quad \begin{cases}
\text{Minimize the objective function, } \sum_{k=1}^{n} (\epsilon_k + \delta_k) \text{, subject to the constraints,} \\[2ex]
A_0 + \sum_{i=1}^{m} A_i z_{ki} - \epsilon_k + \delta_k = f(z_k) \text{ ,} \\
\qquad\qquad\qquad\qquad\qquad\qquad (k = 1 \text{ , } .., n) \\
\epsilon_k \geq 0 \text{ and } \delta_k \geq 0 \text{ ,} \\[1ex]
\text{where the } A_i\text{'s are unrestricted in sign.}
\end{cases}
$$

In order to solve (8) by the simplex method or the revised simplex method, the unknown coefficients, A_i, must be expressed as the difference of two non-negative unknowns, i.e., $A_i = a_i - b_i$, where $a_i \geq 0$, $b_i \geq 0$. With this substitution, (8) becomes:

$$
(9)\quad \begin{cases}
\text{Minimize the objective function, } \sum_{k=1}^{n} (\epsilon_k + \delta_k) \text{, subject to the constraints,} \\[2ex]
a_0 - b_0 + \sum_{i=1}^{m} (a_i - b_i) z_{ki} - \epsilon_k + \delta_k = f(z_k) \text{ ,} \\
\qquad\qquad\qquad\qquad\qquad\qquad (k = 1 \text{ , } .., n) \\
\epsilon_k \geq 0 \text{ , } \delta_k \geq 0 \\[1ex]
\text{and } a_i \geq 0 \text{ , } b_i \geq 0 \text{ for } i = 0 \text{ , } .., m \text{ .}
\end{cases}
$$

The computation time required to obtain the optimum solution of (9) is approximately proportional to n^3, the cube of the number of constraints. Since n will be very large in practice (i.e., of the order of 3000), the computation time will be lengthy and can be reduced by solving the dual problem to (8).

By rewriting (8) in matrix form and applying (6) and (7), the following pair of dual problems is obtained.

<u>Primal problem:</u>

Minimize the objective function, $E = \sum_{k=1}^{n} (\epsilon_k + \delta_k)$, subject to the constraints.

$$\begin{bmatrix} 1 & z_{11} & z_{12} & \cdots & z_{1,m} & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 1 & z_{n1} & z_{n2} & \cdots & z_{n,m} & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ A_m \\ \epsilon_1 \\ \cdot \\ \epsilon_n \\ \delta_1 \\ \cdot \\ \delta_n \end{bmatrix} = \begin{bmatrix} f(z_1) \\ f(z_2) \\ \cdot \\ \cdot \\ f(z_n) \end{bmatrix}$$

where $\epsilon_k, \delta_k \geq 0$, for $k = 1, .., n$, and $A_0, .., A_m$ are unrestricted in sign.

Dual problem:

Maximize the objective function $E' = \sum_{k=1}^{n} f(\hat{z}_k) u_k$, subject to the constraints,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_{11} & z_{21} & \cdots & z_{n1} \\ z_{12} & z_{22} & \cdots & z_{n2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z_{1,m} & z_{2,m} & \cdots & z_{n,m} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix},$$

where the u_i's are unrestricted in sign.

Here the dual problem has $m + 2n + 1$ constraints, so that the computation time is proportional to $(m + 2n + 1)^3$. Hence, the situation is not improved by considering the dual problem directly. However, if a computer program for the simplex algorithm for bounded variables is available, the following equivalent form of the dual problem, with $m + 1$ constraints, can be solved instead with resulting savings in time. [12]

Let $v_i = u_i + 1$ for $i = 1, 2, \ldots, n$.

Maximize the objective function, $E' = \sum_{k=1}^{n} \left\{ f(\vec{z}_k) v_k - f(\vec{z}_k) \right\}$,

subject to the constraints,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_{11} & z_{21} & \cdots & z_{n1} \\ \cdot & \cdot & & \cdot \\ z_{1,m} & z_{2,m} & \cdots & z_{n,m} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} n \\ \sum_{k=1}^{n} z_{k1} \\ \sum_{k=1}^{n} z_{k2} \\ \cdot \\ \cdot \\ \sum_{k=1}^{n} z_{km} \end{bmatrix}$$

where $2 \geq v_i \geq 0$ for $i = 1, \ldots, n$.

V. THE L_∞ APPROXIMATION

In an L_∞ approximation, the function, $P(\vec{z})$, which minimizes the maximum deviation, $\max_{k=1,\ldots,n} |P(\vec{z}_k) - f(\vec{z}_k)|$, is sought.

By introducing a positive number ϵ, the linear programming problem can be stated as:

Minimize ϵ subject to the constraints,

$$\left.\begin{aligned} P(\vec{z}_k) - f(\vec{z}_k) &\leq \epsilon \\ P(\vec{z}_k) - f(\vec{z}_k) &\geq -\epsilon \end{aligned}\right\} \quad \text{for } k = 1, \ldots, n$$

Assuming $P(\vec{z}_k) = A_0 + \sum_{i=1}^{m} A_i z_{ki}$, the problem becomes:

$$(10) \quad \begin{cases} \text{Minimize } \epsilon \text{ (or maximize } -\epsilon\text{) subject to the constraints,} \\ A_0 + \sum_{i=1}^{m} A_i z_{ki} - \epsilon \leq f(\vec{z}_k) \qquad \text{for } k = 1, \ldots, n \text{ and} \end{cases}$$

$$- A_0 - \sum_{i=1}^{m} A_i z_{ki} - \epsilon \leq - f(\vec{z}_k) \tag{10}$$

where the A_i's are unrestricted in sign and ϵ is non-negative.

This problem has 2n constraints. Before solving (10) by the simplex method, the unknown coefficients A_i must be expressed as the difference of two non-negative unknowns, i.e., $A_i = a_i - b_i$. As in the L_1 case, the computation time may be decreased by solving the dual problem. By considering (6) and (7), it can be seen that the dual problem for (10), stated in matrix form, is the following problem with (m + 2) constraints. In deriving the dual problem, all the variables in (10) are considered to be unrestricted in sign, since the form of the constraints in (10) ensures that ϵ is non-negative.

(11)

Minimize the objective function, $\sum_{k=1}^{n} [f(\vec{z}_k)u_k - f(\vec{z}_k)v_k]$, subject to the constraints,

$$\begin{bmatrix} 1 & \cdot\cdot\, 1 & 1 & \cdot\cdot\, 1 \\ 1 & \cdot\cdot\, 1 & -1 & \cdot\cdot\, -1 \\ z_{11} & \cdot\cdot\, z_{n1} & -z_{11} & \cdot\cdot\, -z_{n1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_{1,m} & \cdot\cdot\, z_{n,m} & -z_{1,m} & \cdot\cdot\, -z_{n,m} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \\ v_1 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

and $u_k \geq 0$, $v_k \geq 0$, for $k = 1, \ldots, n$.

Similar techniques can be used to determine the polynomial, $P(\vec{z})$ of a given form which approximates $f(\vec{z})$ while keeping the deviation, $|P(\vec{z}_k) - f(\vec{z}_k)|$, at the k-th point within a predetermined tolerance ϵ_k, for $k = 1, \ldots, n$. Assuming that $P(\vec{z}_k) = A_0 + \sum_{i=1}^{m} A_i z_{ki}$, and introducing $\eta_k = 1/\epsilon_k$, for $k = 1, \ldots, n$, the problem becomes that of determining $A_0, \ldots, A_m$ so that $\max\limits_{k=1,\ldots,n} \eta_k \left| A_0 + \sum\limits_{i=1}^{m} A_i z_{ki} - f(\vec{z}_k) \right|$ is minimized. If this minimum value does not exceed 1, then the deviation of $P(\vec{z})$ from $f(\vec{z})$ is within the desired tolerance at each data point. Defining $A_0 + \sum\limits_{i=1}^{m} A_i z_{ki} - f(\vec{z}_k) = d_k - e_k$, where $d_k \geq 0$, $e_k \geq 0$, and introducing the non-negative variable t, the problem can then be expressed as the following linear programming problem.

$$
(12)\quad \left\{
\begin{array}{ll}
\text{Minimize the objective function, } d_1 + e_1 + t, \text{ subject to} & \\
\text{the constraints,} & \\
d_k - e_k - \eta_k A_0 - \eta_k \sum\limits_{i=1}^{m} A_i z_{ki} = \eta_k f(\vec{z}_k), & \\
d_1 + e_1 + t - d_k - e_k \geq 0 & (k = 1, \ldots, n) \\
d_k \geq 0, \quad e_k \geq 0 & \\
t \geq 0, \text{ and } A_0, \ldots, A_m \text{ are unrestricted in sign.} &
\end{array}
\right.
$$

If the objective function, $d_1 + e_1 + t$, for the optimum solution exceeds 1, then there is no approximating function of the given form satisfying the given tolerances. In solving (12), the coefficients A_i should be expressed as $A_i = a_i - b_i$, where $a_i \geq 0$, $b_i \geq 0$. The problem has (2n-1) constraints. There appears to be no saving in computation time in solving the dual problem.

An alternative formulation of the problem follows:

Maximize γ subject to the conditions,

$|P(\vec{z}_k) - f(\vec{z}_k)| \leq \epsilon_k - \gamma$,

i.e.,

$$
\begin{array}{ll}
\gamma + P(\vec{z}_k) \leq \epsilon_k + f(\vec{z}_k) & \\
\text{and} & (k = 1, \ldots, n) \\
\gamma - P(\vec{z}_k) \leq \epsilon_k - f(\vec{z}_k) &
\end{array}
$$

If the maximum value of γ is non-negative, then a polynomial of the desired form satisfying the given tolerances has been found. Assuming $P(\vec{z}_k) = A_0 + \sum\limits_{i=1}^{m} A_i z_{ki}$, the problem then becomes:

$$
(13)\quad \begin{cases} \text{Maximize } \gamma \text{ subject to the constraints} \\ \gamma + A_0 + \sum_{i=1}^{m} A_i z_{k\,i} \le \epsilon_k + f(\vec{z}_k) \text{ and} \\ \gamma - A_0 - \sum_{i=1}^{m} A_i z_{k\,i} \le \epsilon_k - f(\vec{z}_k) \\ \text{where } \gamma, A_0, \ldots, A_m \text{ are unrestricted in sign.} \end{cases} \qquad (k = 1, \ldots, n)
$$

The dual problem to (13) has only (m + 2) constraints, and hence should be solved in preference to (13).

$$
(14)\quad \begin{cases} \text{Minimize } \sum_{k=1}^{n} \left\{ [\epsilon_k + f(\vec{z}_k)]u_k + [\epsilon_k - f(\vec{z}_k)]v_k \right\} \text{ subject to the constraints,} \\ \begin{bmatrix} 1 & \cdot\cdot & 1 & 1 & \cdot\cdot & 1 \\ 1 & \cdot\cdot & 1 & -1 & \cdot\cdot & -1 \\ z_{11} & \cdot\cdot & z_{n1} & -z_{11} & \cdot\cdot & -z_{n1} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ z_{1,m} & \cdot\cdot & z_{n,m} & -z_{1,m} & \cdot\cdot & -z_{n,m} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \\ v_1 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \\ \text{and } u_k \ge 0,\ v_k \ge 0, \text{ for } k = 1, \ldots, n. \end{cases}
$$

VI. SPEED OF CONVERGENCE

The special structure of the right-hand vector in the constraint equations in (11) and (14) causes slow convergence of the simplex algorithm. In this section, the steps of the algorithm are analyzed for this special case, and a transformation of the original problem, which increases the speed of convergence of the algorithm, is discussed. The notation in this section is that which is frequently used in discussions of the simplex method and has no connection with the notation of the preceeding sections.

Let the following array denote the simplex tableau, where the d_k's are shadow prices ($d_k = z_k - C_k$), the v_i's are activity levels, and z is the value of the objective function.

z	d_1	$\cdots$ d_{2n}
v_1	x_{11}	$\cdots$ $x_{1,2n}$
v_2	x_{21}	$\cdots$ $x_{2,2n}$
$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$
v_{m+1}	$x_{m+1,1}$	$\cdots$ $x_{m+1,2n}$

If the k-th column of the tableau is introduced into the basis, then the column in the tableau corresponding to the j-th column in the basis is eliminated from the basis, where j is given by

$$v_j/x_{jk} = \min_{\substack{i=1,\ldots,m+1 \\ x_{ik} > 0}} (v_i/x_{ik}) = \theta \ .$$

The change in the objective function is $-\theta d_k$. Since d_k is positive and $\theta \geq 0$, the objective function is decreased only if $\theta > 0$. Initially, the vector of activity levels is the column vector $(1,0,\ldots,0)$. Hence the value of the objective function will be improved initially only if $x_{ik} \leq 0$ for $i = 2, 3, \ldots, m+1$ and $x_{1k} > 0$. If the problem has a bounded optimum solution, then $x_{1k} > 0$ if $x_{ik} \leq 0$ for $i = 2, \ldots, m+1$. Therefore the probability that $\theta > 0$ is 2^{-m} when the vector of activity levels is the column vector $(1,0,\ldots,0)$ and the signs of the x_{ij}'s are assumed to be random. If $x_{ik} > 0$ initially, for some $i > 1$, then the new vector of activity levels is still $(1,0,\ldots,0)$. However, when $x_{ik} \leq 0$ for $i = 2, \ldots, m+1$, then the new vector of activity levels will contain at least two positive elements. By the same argument, the probability that the objective function will be decreased at the next iteration is at least 2^{-m+1}. The same argument is repeated until the optimum solution is obtained. The expected number, N, of iterations required to reach the optimum solution will thus satisfy the inequality,

$$2^m \leq N \leq 2^{m-1} + \cdots + 2^0 < 2^{m+1} \quad ,$$

in the special case where the right-hand vector is $(1,0,\ldots,0)$. This is much larger than the estimated $3(m+1)/2$ or $2(m+1)$ iterations usually required to reach an optimum solution.

Before solving the problem by the simplex method, a transformation can be applied to the problem in the following manner in order to eliminate the zero elements in the right-hand vector which slow the convergence. The original problem is to minimize CX subject to the constraints, $AX = B$, and $X \geq 0$, where b is the column vector $(1,0,..,0)$. Let D be the $(m+1) \times (m+1)$ transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 & . & . & 0 \\ 1 & 1 & 0 & . & . & 0 \\ 1 & 0 & 1 & . & . & 0 \\ . & . & . & & & . \\ . & . & . & & & . \\ 1 & 0 & 0 & . & . & 1 \end{bmatrix}$$

Consider the transformed set of constraints, $DAX = Db$. Each component of the right-hand vector, Db, is 1. Since D is non-singular, the problem of minimizing CX subject to the constraints, $DAX = Db$, and $X \geq 0$ has the same solution as the original problem. The solution, $W_0 = C_0 B^{-1}$, of the original problem, where B is the optimum basis, can be readily obtained from the solution W_0' of the transformed problem since $(DB)^{-1}$ is the optimum basis of the transformed problem. Hence, $W_0' = C_0 (DB)^{-1} = C_0 B^{-1} D^{-1} = W_0 D^{-1}$ and therefore $W_0 = W_0' D$. The final simplex tableaux for both problems are the same, since if P_k represents the k-th column of the final tableau for the original problem is $B^{-1} P_k$, and that for the transformed problem is $(DB)^{-1} (DP_k)$.

VII. NUMERICAL RESULTS

This section contains the results of preliminary tests run at UNC. In these tests, only linear terms were included in the approximating polynomial for χ.

$$P(\vec{z}_k) = A_1 + A_1 x_k + A_2 y_k + A_3 \dot{x}_k + A_4 \dot{y}_k + A_5 (F/m)_k + A_6 t_k \ .$$

Five evenly spaced data points on each of five trajectories were used. Difficulty in obtaining an optimum solution was experienced in the early stages of the tests because of the wide range in the magnitudes of the variables x, y, $\dot{x}$, etc. However, after scaling the elements of the constraint matrix so that all elements were of the same order of magnitude, the optimum solution was successfully obtained.

L_1 approximation: The results of this calculation are included for comparison with the L_∞ case.

i	A_i
0	-203.786
1	.236515
2	.0576878
3	4.75189
4	-37.0782
5	2663.88
6	-.724890

$$\sum_{k=1}^{25} |P(\vec{z}_k) - \chi(\vec{z}_k)| = 5.10843$$

$$\max|P(\vec{z}_k) - \chi(\vec{z}_k)| = .974130 .$$

L_∞ approximation: Both the primal formulation (10) and the dual formulation (11) were tested. For the dual problem the effect of the transformation discussed in section VI was tested. The values of the coefficients, A_i , computed in the three different ways are shown in the following table.

	Primal	Dual	
		transformed	untransformed
A_0	100.382	121.861	23.918
A_1	.203273	.183696	.187623
A_2	.0033899	-.000799465	-.00108843
A_3	-25.4105	-27.4675	3.71590
A_4	-56.1931	-51.9564	-65.4972
A_5	4511.03	4189.05	5042.98
A_6	-.291315	-.219835	-.303204
A_7 (ϵ)	.7823	.7823	.7710
Value of objective function.	.7823	.7823 (.7882*)	.7823 (7823*)
No. of iterations.	not available	21	30
Max $\|P-\chi\|$	.7823	.8627	1.709989

*This value was computed by $x_0 = (B^{-1}b)_0$.

There are several points to be observed:

1) There is quite a large discrepancy between the three solutions, probably caused by the accumulation of round-off errors in the performance of the single precision, floating point arithmetic used in the linear programming routine.

2) The values of the objective function agree up to four decimal places for the three trials. This value should be the least upper bound for $\max_k |P(\vec{z}_k) - \chi(\vec{z}_k)|$. However, for the dual formulation of the problem, $\max_k |P(\vec{z}_k) - \chi(\vec{z}_k)| > \epsilon$. For the untransformed dual problem, the discrepancy is large.

3) The number of iterations required to obtain the optimum solution of the transformed dual problem was smaller than the number required for the untransformed dual problem, as expected. However, the number of iterations for the untransformed problem was smaller than expected. This was probably caused by the failure of the elements of the tableau to satisfy the random sign hypothesis in this small test. In a realistic case in which the number of constraints is large, the hypothesis of random signs will be more nearly satisfied and the relative decrease in computation time for the transformed problem should be larger.

In order to reduce the effect of round-off errors in the calculation of B^{-1} for the dual problem, B^{-1} was computed iteratively by

$$B_{i+1}^{-1} = B_i^{-1}(2I - BB_i^{-1}) \quad ,$$

where B_i^{-1} is the i-th approximation to B^{-1} and I is the identity matrix. In this experiment, B_0^{-1} was the matrix obtained by the simplex method. Only one iteration was carried out. The set of coefficients in $P(\vec{z})$ were computed from $W_0 = C_0 B_1^{-1}$. The result is given in the following table. The results of the previous calculation for the primal problem are included for comparison.

	Primal	Dual	
		transformed	untransformed
A_0	100.382	100.380	100.374
A_1	.203273	.203284	.203285
A_2	.0033899	.0033922	.0033922
A_3	-25.4105	-25.4116	-25.4120
A_4	-56.1931	-56.1943	-56.1954
A_5	4511.03	4511.14	4511.22
A_6	-.291315	-.291353	-.291349
A_7 (ϵ)	.7823	.7801	.7823
Value of objective function	.7823	.7823 (.7773)	.7823 (.7823)
$\mathrm{Max}\lvert P-\chi\rvert$	.7823	.7879	.7829

Improvement in accuracy is apparent. The three sets of coefficients obtained independently agree to four figures. $\max\limits_k |P(\vec{z}_k) - \chi(\vec{z}_k)|$ is much closer to the value of the objective function. However, the following unexplained points were observed:

1) $B_0^{-1}B$ is very close to I but BB_0^{-1} differs considerably from I.

2) $B_1^{-1}B$ has a greater deviation from I than does $B_0^{-1}B$.

3) The accuracy of $A_0, \ldots, A_6$ seems to be improved, but that of ϵ decreased.

VIII. DIFFERENT CHOICES OF PIVOTAL ELEMENTS

In the progress report, [6], several criteria for the choice of pivotal elements were discussed. The Greatest Absolute Ascent Method and the Modified Gradient Method were tested in addition to Dantzig's usual criterion. The accumulation of round-off errors appeared to be greater in the first two methods than in Dantzig's method and led to incorrect decisions. For this reason the alternative criteria do not appear to possess the merit claimed by Quandt and Kuhn. [9] This difficulty might be removed by using double precision arithmetic or by improving the inverse of the basis by some iterative method after several iterations of the algorithm.

IX. CONCLUSIONS

L_1 and L_∞ approximation problems, or modification of them, can be formulated as linear programming problems. Comparison of this approach with the least squares procedure is difficult. The choice of a method depends upon the requirements and characteristics of the approximation problem. For example, if the deviation of the approximating function from $f(\vec{z})$ is to be within a prescribed tolerance, then the linear programming approach is straightforward and effective. An expected merit of the use of linear programming is the accuracy or stability of the solution. In the least square method an ill-conditioned matrix of normal equations often causes trouble. Round-off error may cause inaccuracies in the linear programming approach. In particular, inaccuracies may arise in the process of obtaining the solution of the primal problem from that of the dual. This difficulty can be removed more easily in the linear programming approach than in the least squares method. The scaling problem is less complex in the linear programming case than in the least squares case. However, the computation time for obtaining the approximating function is at least as long by linear programming as by least squares.

The major results are summarized as follows:

1. L_1 or L_∞ approximations of $f(\vec{z})$ by a function

$$P(\vec{z}) = A_0 + \sum_{i=1}^{m} A_i P_i(W_{ki}) + \sum_{i>j}^{m} A_{ij} P_{ij}(W_{ki}, W_{kj}) + \cdots$$

   can be formulated as linear programming problems.

2. For the multivariate approximation problem the dual program can usually be solved more quickly than the primal program. The solution for the primal program can easily be obtained from that of the dual program.

3. The dual program has a special structure which may cause slow convergence to an optimal solution. Therefore, the constraint matrix should be transformed (an example of such a transformation is in VI) before the problem is solved.

4. The data of the original simplex tableau should be scaled so that the data are of the same order of magnitude.

5. Single precision arithmetic seems to be insufficient for obtaining the solution of the primal problem from that of the dual problem. A large error may occur in $W_0 = C_0 B^{-1}$ even if an accurate solution for the dual program is available. Therefore the inverse matrix should be computed in double precision arithmetic. The reinversion of the current basis is also advisable.

REFERENCES

1. Ackoff, R. L., editor. Progress in Operations Research. Operations Research Society of America, Publication in Operations Research No. 5, 1961.

2. Gass, S. I. Linear Programming. McGraw-Hill, 1958.

3. Goldman, A. J. and A. W. Tucker. "Theory of Linear Programming." Linear Inequalities and Related Systems, Annuals of Mathematics No. 38, Princeton, 1956.

4. Hanson, J. W. and S. M. Hubbard. "Linear Programming Applied to Guidance Function Fitting." Progress Report No. 1 on Studies in the Fields of Space Flight and Guidance Theory, MSFC Report No. MTP-AERO-61-91, Dec. 18, 1961.

5. Hoelker, R. F. and W. E. Miner. "Introduction to the Concept of the Adaptive Guidance Mode." MSFC Aeroballistics Internal Note No. 21-60, Dec. 28, 1960.

6. Hubbard, S. M. and S. Suzuki. "Linear Programming Applied to Guidance Function Fitting." Progress Report No. 2, MSFC Report No. MTP-AERO-62-52, June 26, 1962.

7. Kelley, J. E. Jr. "An Application of Linear Programming to Curve Fitting." J.S.I.A.M., Vol. 6, No. 1, March, 1958.

8. Miner, W. E. and D. H. Schmieder. "Status Report on the Adaptive Guidance Mode." MSFC Report No. MTP-AERO-61-18, March 13, 1961.

9. Quandt, R. E. and H. W. Kuhn. "On Some Computer Experiments in Linear Programming." Presented at ISI Meeting, Paris, August, 1961.

10. Swartz, R. V. and R. J. Vance. "Adaptive Guidance Mode Studies." Chrysler Corporation Missile Division, Technical Note AAN-TN-12-61, August 30, 1961.

11. Wagner, H. M. "Linear Programming Techniques for Regression Analysis." J.A.S.A., Vol. 54, March, 1959.

12. Wagner, H. M. "The Dual Simplex Algorithm for Bounded Variables." Naval Research Logistics Quarterly 5, No. 3, 1958.

13. Ward, L. E. Jr. "Linear Programming and Approximation Problems." Amer. Math. Monthly, Vol. 68, Jan. 1, 1961.

APPROVAL

PROGRESS REPORT NO. 3

on

STUDIES IN THE FIELDS OF

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Sponsored by

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